Real Time Systems and Control Applications

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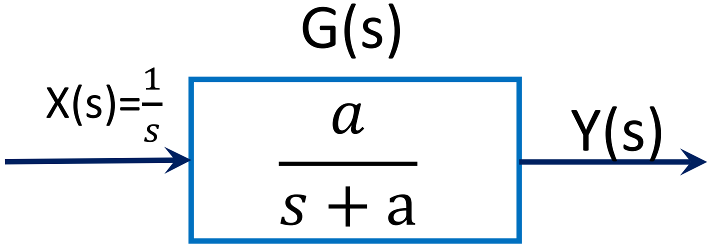
First Order Systems

Second Order Systems

Time Constant of First Order Systems

- First order system



- The output of a general first order system to a step input:

$$Y(s) = X(s)\,G(s) = \frac{a}{s\,(s+a)}$$

- This results in a time domain output given by: $\mathrm{y}(t) = 1 - \; e^{-at}$, where parameter a is the only parameter that affects the output.

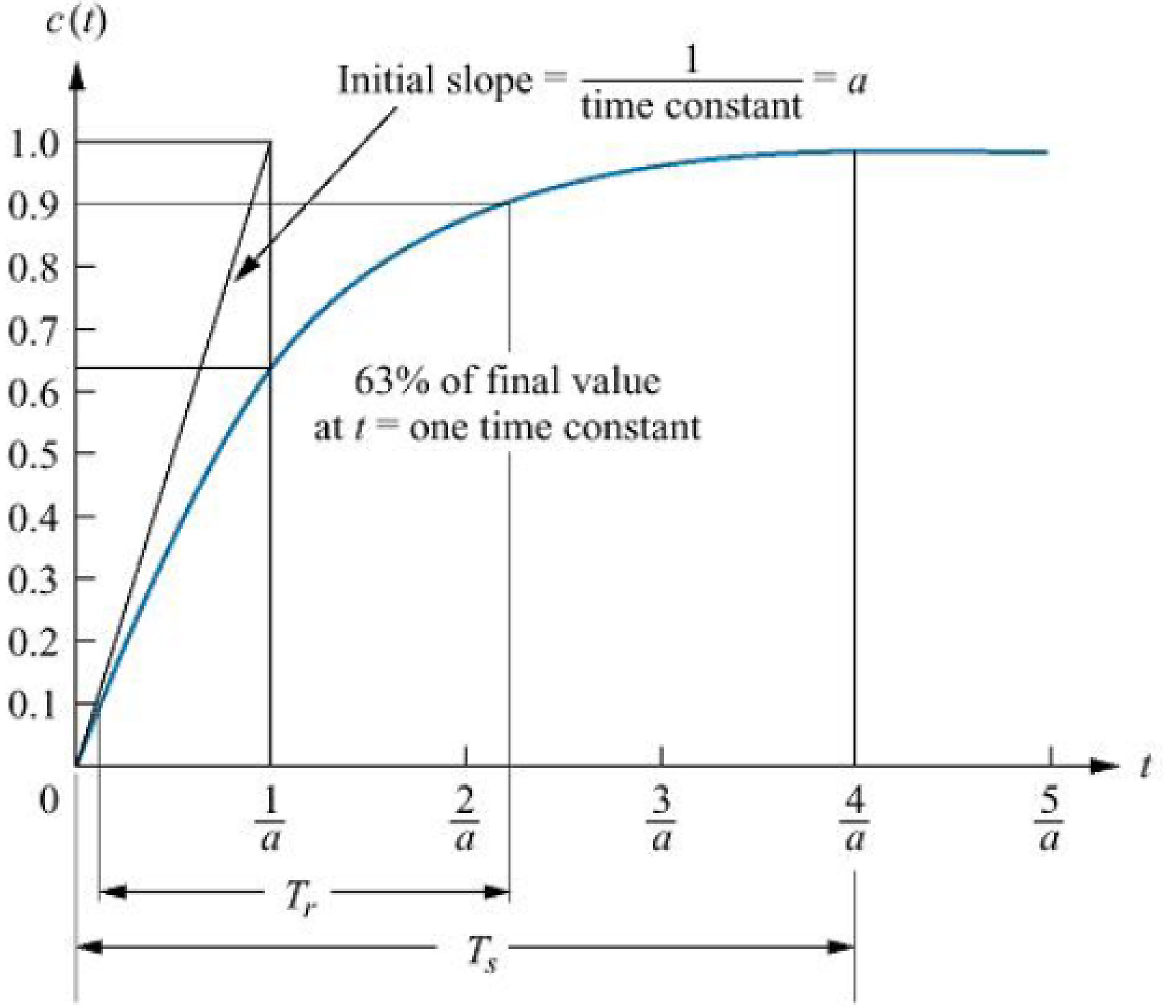
$$t = \frac{1}{a}, \mathrm{y}(t) = 0.63.$$

$\frac{1}{a}$ is the time constant of the response, and is the time it takes for the step response to rise to 63% of its final value.

Response in Time Domain

- Rise Time (T_r): time for the waveform to go from 0.1 to 0.9 of its final value. For first order systems: $T_r = \frac{2.2}{a}$

- Settling Time (T_s): time for the response to reach and stay within 2% of its final value. For first order systems: $T_s = \frac{4}{a}$



Another First Order Response Example

- Give $G(s) = \frac{1}{s+a}$, what is the time constant, rise time, and settling time?
  - $\mathrm{Y}(s) = \frac{1}{a}\left(\frac{1}{\mathrm{s}} - \frac{1}{s+a}\right) \Rightarrow y(t) = \frac{1}{a}(1 - e^{-at})$

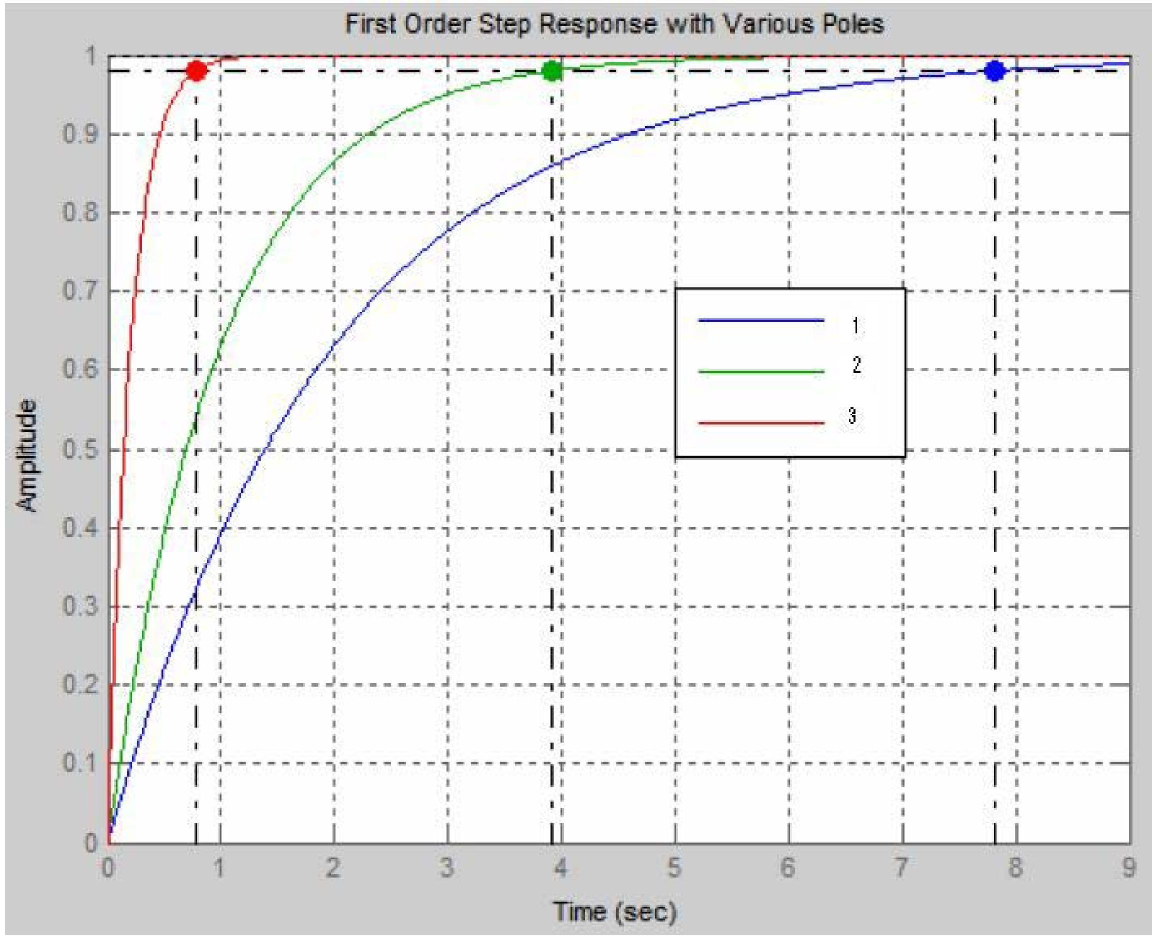
  Solution:
  - Time constant τ makes $y(\tau) = \frac{1}{a} * 0.63$, so $\tau = \frac{1}{a}$
  - Rise time $t_r = t_2 - t_1$, where t_2 makes $y(t_2) = \frac{1}{a} * 0.9$ and t_1 makes $y(t_1) = \frac{1}{a} * 0.1$, hence $t_r = \frac{2.3-0.1}{a} = 2.2 * \frac{1}{a}$
  - Settling time t_s makes $y(t_s)$=0.98$\frac{1}{a}$, hence $t_s \cong \frac{4}{a}$

More Example

The figure shows the response of three first order systems having transfer function $\frac{K}{s+a}$, where the values of K are different for the three systems.

Answer the following questions:

- 1. Which of the three curves (1, 2, 3) represents a system with the lowest time constant?
- 2. The big dots on the three graphs represent the time when the response settles within 2% of the final value. Find the transfer function for each of the three systems.



Solution

- The settling time for first order systems is given by $T_s=\frac{4}{a}$.

- From the figure, the values of T_s are 7.8, 3.9 and 0.8 respectively, so the value a for 3 systems are roughly 0.5, 1.0 and 5 respectively.

Since the steady state value of each system is 1, so K=a. Therefore the transfer function of the systems are $\frac{0.5}{s+0.5}$, $\frac{1}{s+1}$, and $\frac{5}{s+5}$.

Second Order Systems

- Most real-world systems are not first order systems. A general second order system defined by the transfer function:

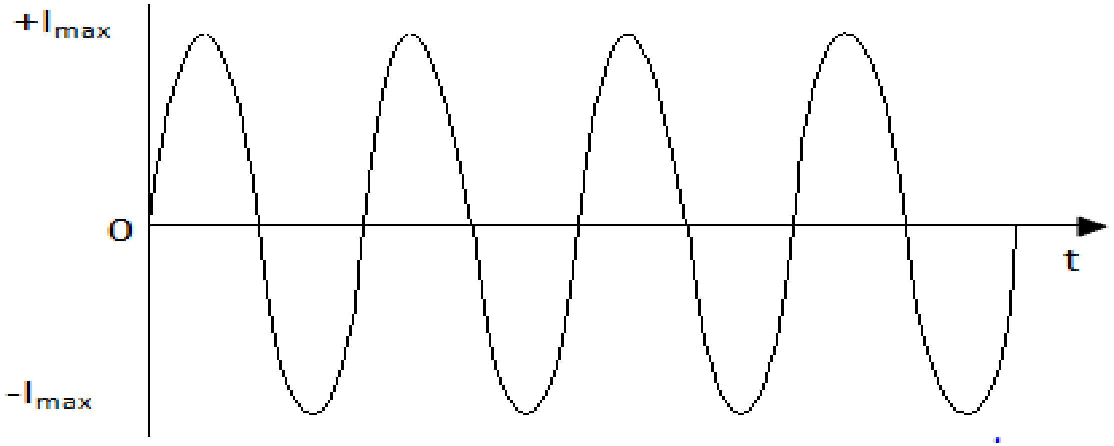
$$G(s) = \frac{b}{s^2 + as + b}$$

- Find the poles of this transfer function to examine the behaviour of the output response. Using quadratic formula:

$$s_1, s_2 = \frac{-a \pm \sqrt{a^2 - 4\mathrm{b}}}{2}$$

A Special Case (a=0)

- If a = 0, the transfer function is $\mathrm{G(s)} = \frac{b}{s^2+b}$, and the poles will have only imaginary part $\pm jw$ and by definition the **natural frequency** $w_n = \sqrt{b}$ is the frequency of oscillation of this system.



Undamped Case

Damping Coefficient ζ, when a is not zero

- The complex poles have a real part $\sigma = \frac{-a}{2}$.
- The magnitude of σ is called the exponential decay frequency, and w_n the natural frequency. We define the *Damping Ratio* or *Damping Coefficient*, ζ as

$$\zeta = \frac{Exponential\ decay\ frequency}{Natural\ frequency}$$

$$\therefore\ \zeta = \frac{|\sigma|}{w_n} = \frac{\frac{a}{2}}{w_n} \quad so\ that\ \ a = 2\zeta\, w_n$$

General Second Order Transfer Function

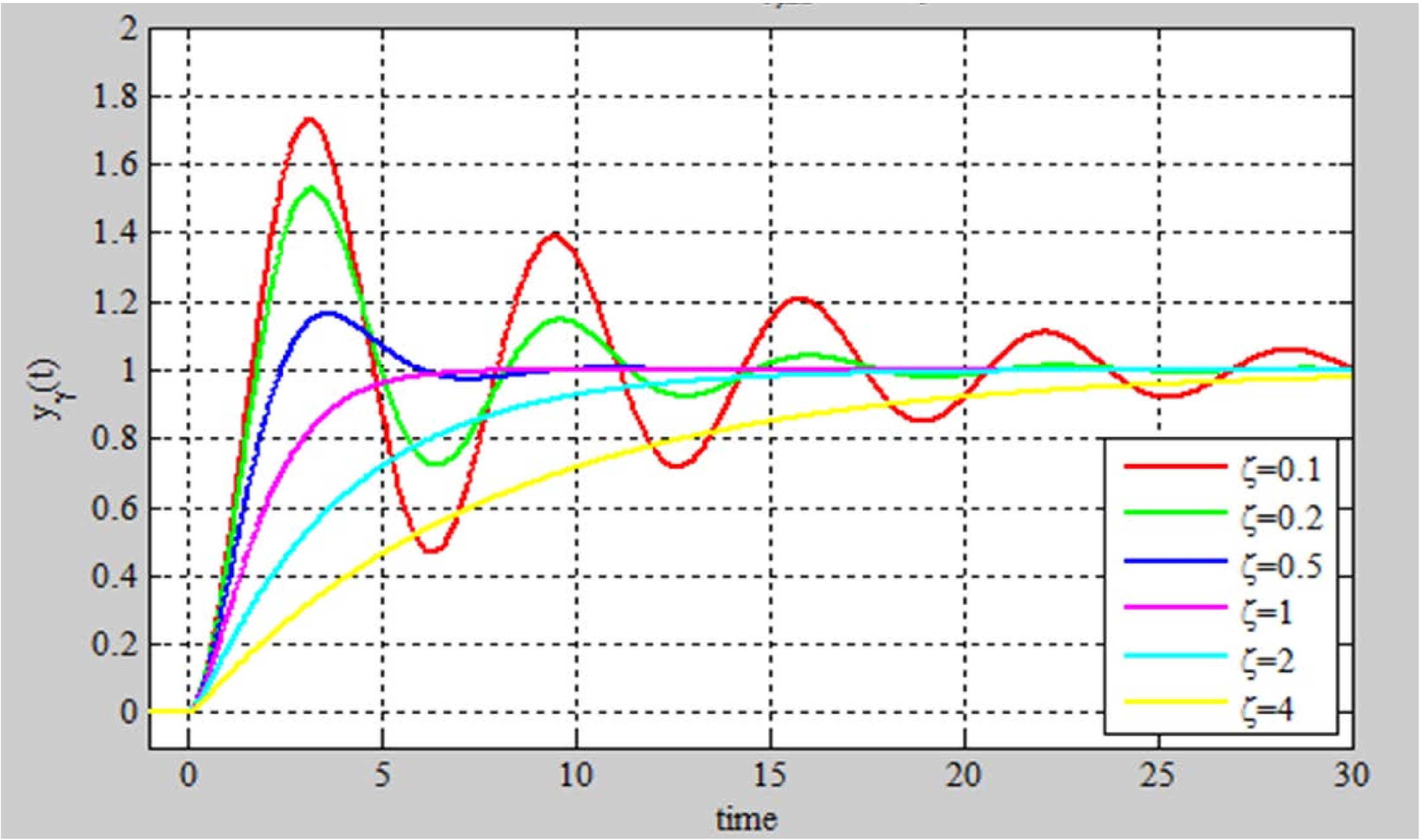
- The general second order transfer function can now be written as:

$$\mathrm{G(s)} = \frac{{w_n}^2}{s^2 + 2\zeta\, w_n\, s + \ {w_n}^2}$$

$$s_1, s_2 = -\zeta\, w_n \pm w_n\sqrt{\zeta^2 - 1}$$

- From above we can examine the effect of parameter ζ on the output of a second order system.

Effect Of Parameter ζ



Summary of Observations

- Two imaginary poles at $\pm j\omega_n$: ζ = 0 (**undamped**)

Natural response: undamped sinusoid of frequency ω_n equal to the imaginary part of the poles. Or $c(t) = A\cos(\omega_n t - \varphi)$

- Two complex poles at $\sigma_d \pm j\omega_d$: 0 < ζ < 1 (**underdamped**)

Natural response: Underdamped response in the form of sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. Or $c(t) = A\, e^{(-\sigma_d)t}\cos(\omega_d t - \varphi)$, where $w_d = w_n\sqrt{1 - \zeta^2}$.

Continued…

- Two real poles at σ_1: $\zeta = 1$ (**critically damped**)

Natural response: critically damped system has the time domain response as:

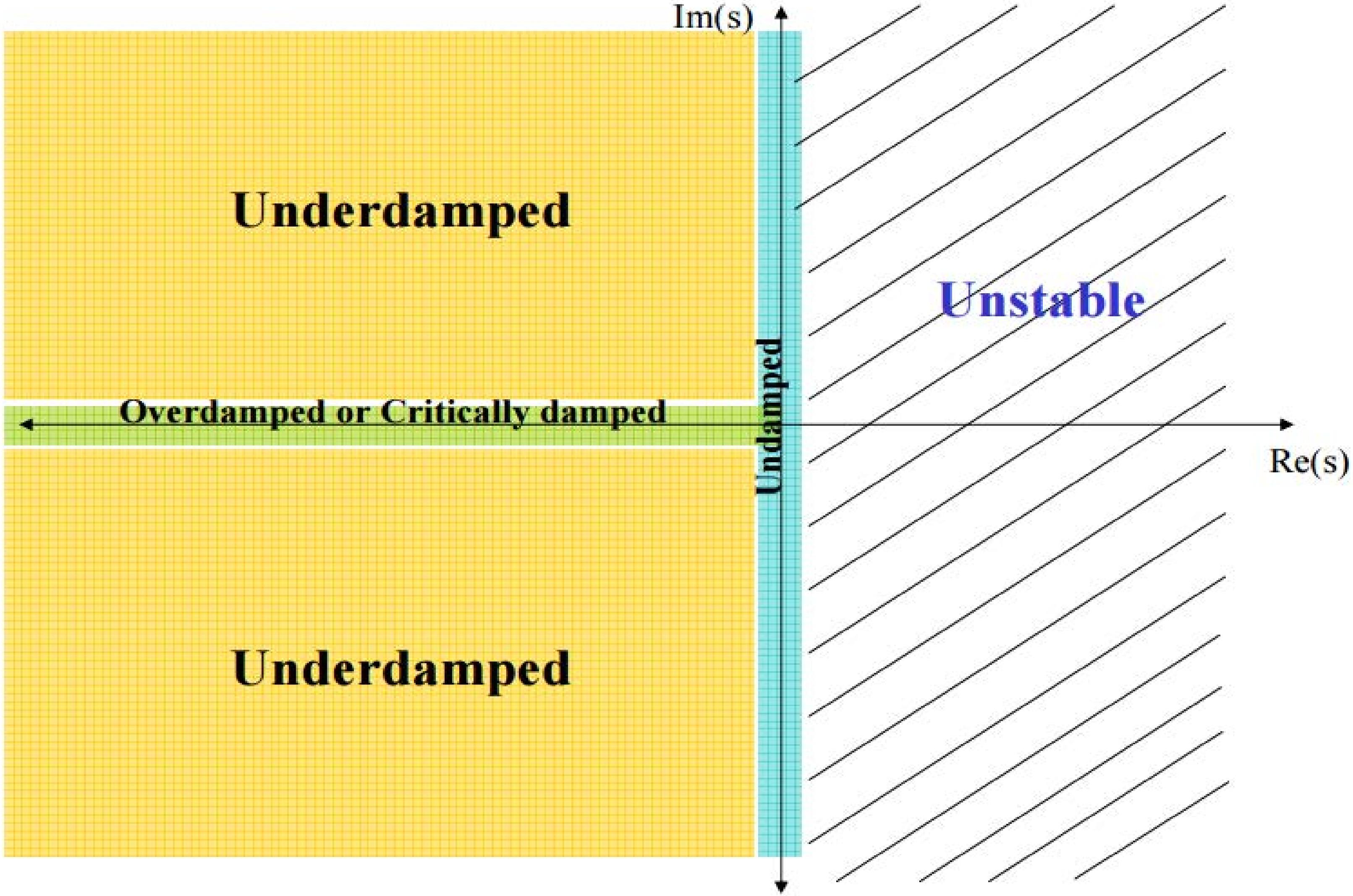
$$c(t) = Kte^{\sigma_1 t}$$

- Two real poles at σ_1 and σ_2: $\zeta > 1$ (**overdamped**)

Natural response: overdamped with two exponentials having time constants equal to the reciprocal of the pole locations. Or

$$c(t) = K(e^{\sigma_1 t} + e^{\sigma_2 t})$$

Second Order Impulse Response



Underdamped Second Order Step Response

- The general Transfer Function of a second order system is:

$$\mathrm{G(s)} = \frac{{w_n}^2}{s^2 + 2\zeta\, w_n\, s + \ {w_n}^2}$$

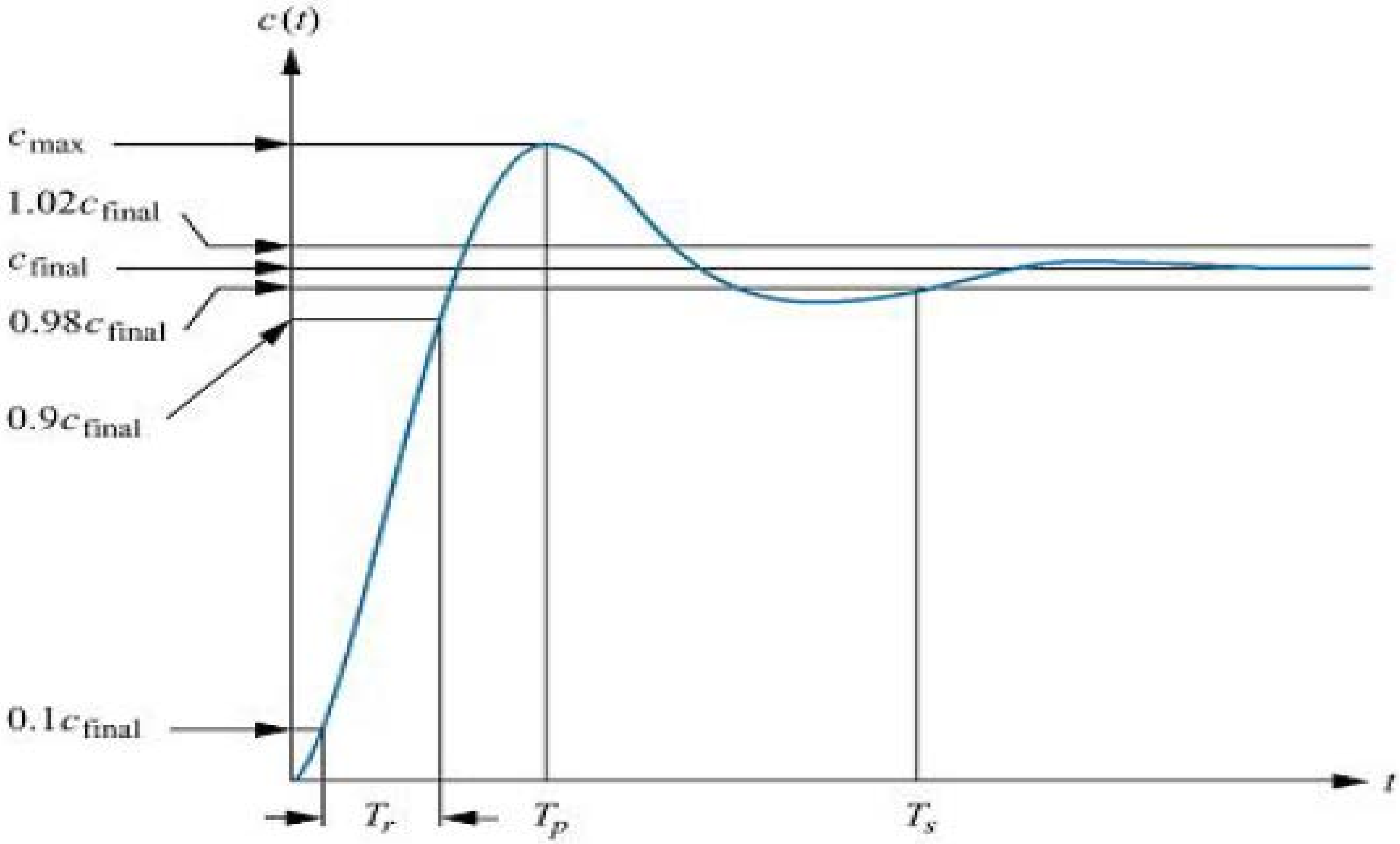
- Consider response for a step input. The transfer function of response C(s) is given by:

$$\mathrm{C(s)} = \frac{{w_n}^2}{s(s^2 + 2\zeta\, w_n\, s + \ {w_n}^2)}$$

- Taking the inverse LT to get response in time domain results in:

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\, w_n t} \cos(\sqrt{1-\zeta^2}\omega_n t + \varphi)$$

where $\varphi = tan^{-1}\,(\frac{\zeta}{\sqrt{1-\zeta^2}})$.


$c(t)$
c_{max}
$1.02c_{final}$
c_{final}
$0.98c_{final}$
$0.9c_{final}$
$0.1c_{final}$
t
T_r
T_p
T_s

Peak Time, Tp

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} \cos(\sqrt{1-\zeta^2}\omega_n t + \varphi)$$

$$\varphi = tan^{-1} \left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

The time required to reach the first or maximum peak. This can be found by differentiating c(t), and equating to zero which gives:

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Because c'(t)= $-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} \sin\left(\sqrt{1-\zeta^2}\omega_n t + \varphi\right)\sqrt{1-\zeta^2}\omega_n -$

$$\frac{1}{\sqrt{1-\zeta^2}}(-\zeta w_n) e^{-\zeta w_n t} \cos\left(\sqrt{1-\zeta^2}\omega_n t + \varphi\right) = 0$$

$$\therefore tan\left(\sqrt{1-\zeta^2}\omega_n t + \varphi\right) = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

Therefore, $\sqrt{1-\zeta^2}\omega_n t = \pi$ ➔ $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

Percent Overshoot, % OS

The amount that the waveform overshoots the steady state of final value at peak time, expressed as percentage of steady state value: %OS $= \frac{c_{max} - c_{final}}{c_{final}} \times 100$, where $c_{final} = 1$ and $c_{max} = c(T_p)$. Substituting the expression for $c(T_p) = 1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$ in previous subsection and some manipulation results in: %OS $= e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100.$

Note that %OS is a function of ζ, the damping ratio only. The above expression gives an expression for ζ in terms of %OS.

$$\zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + ln^2(\frac{\%OS}{100})}}$$

Finding T_p and %OS From Transfer Function

- Given a transfer function $G(s) = \frac{100}{s^2+15s+100}$, find T_p and %OS.

Solution: $\omega_n = \sqrt{100} = 10$ and ζ = $\frac{\frac{a}{2}}{\omega_n}$ =15/20 = 0.75.

$$\therefore T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.475\ s$$

And finally, $\%OS = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100 = e^{-\frac{0.75\pi}{\sqrt{1-0.75^2}}} \times 100 = 2.838\%.$

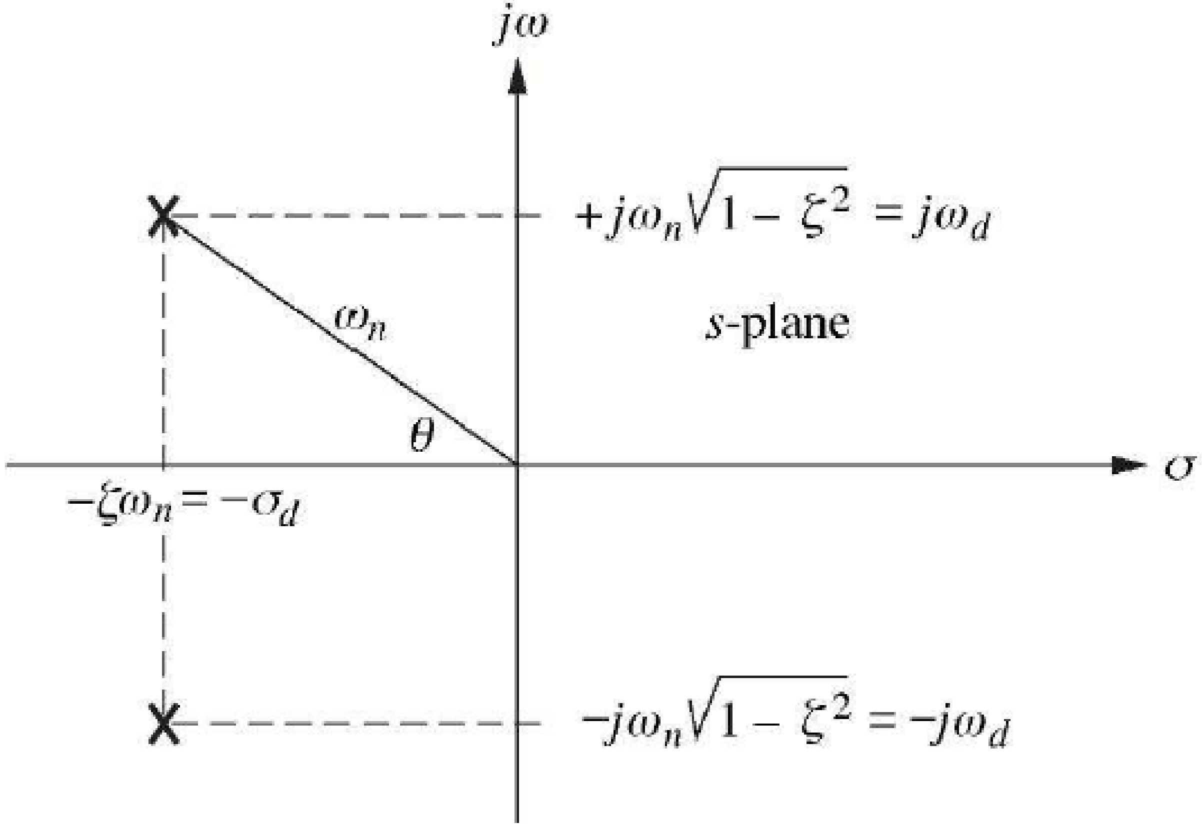
How are T_p And T_s Related To Location of Poles

$$G(s) = \frac{{w_n}^2}{s^2 + 2\zeta\, w_n\, s + \ {w_n}^2}$$

$$s_1, s_2 = -\zeta\, w_n \pm jw_n\sqrt{\zeta^2 - 1}$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1 - \zeta^2}}$$

$$T_s \cong \frac{4}{\zeta\, \omega_n}$$

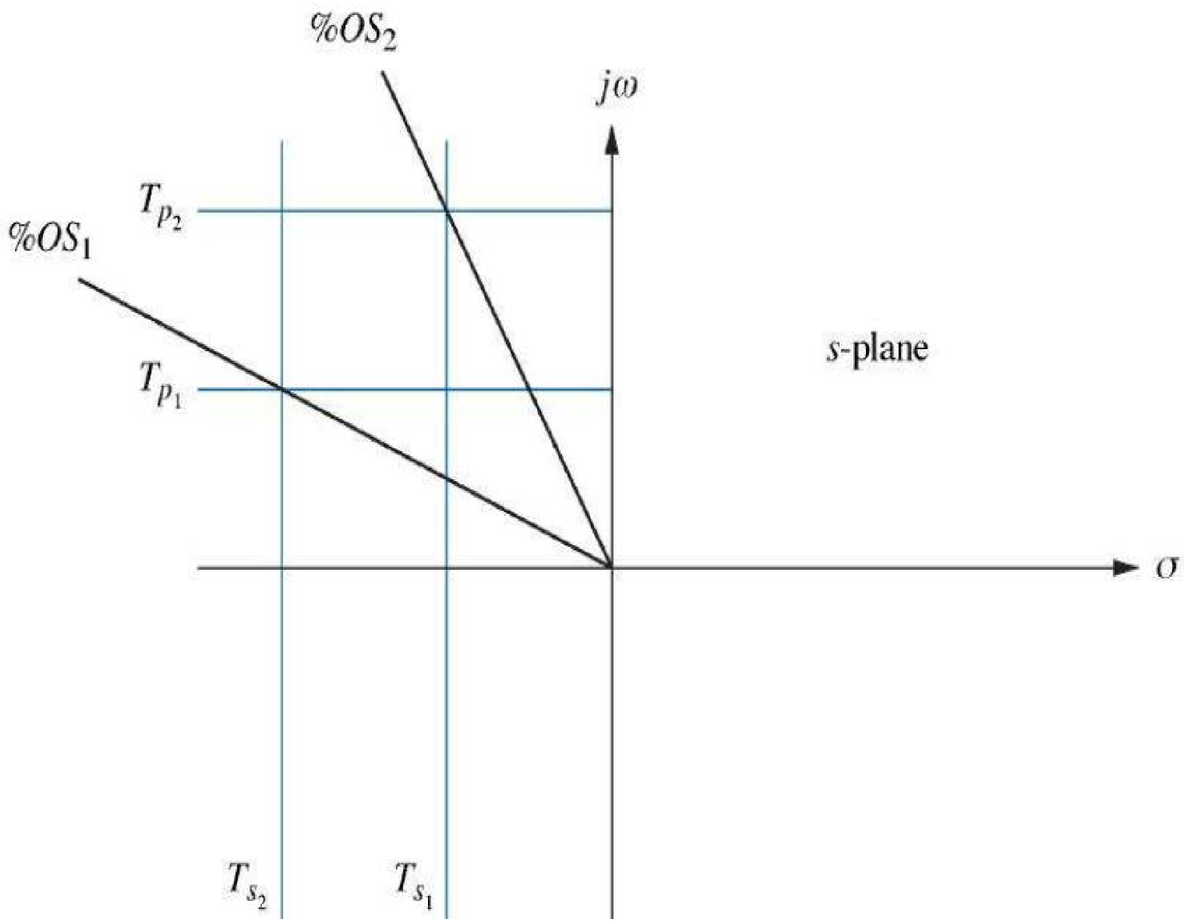


Poles of a second order underdamped system

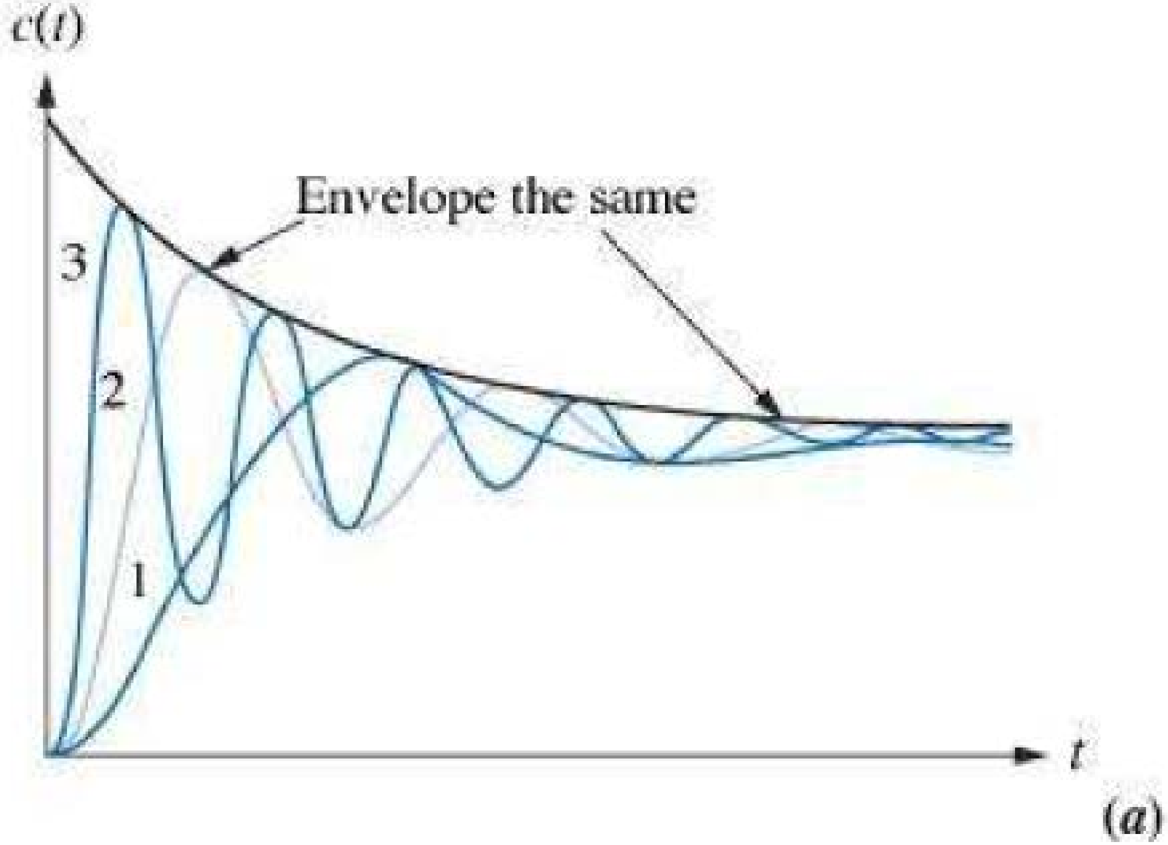
Lines of Constants T_p, T_s and %OS on s-Plane

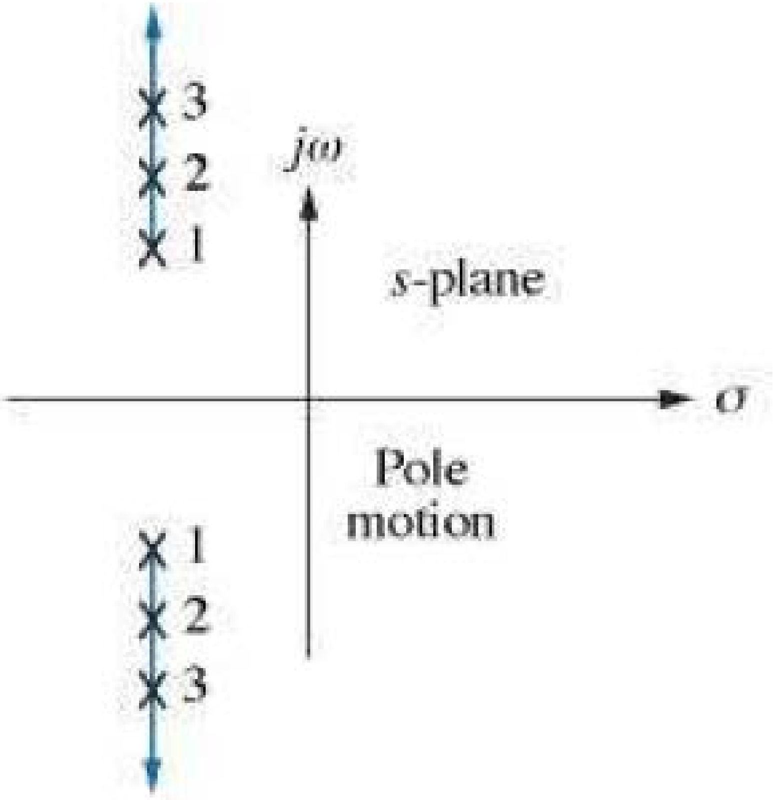
- Note that horizontal lines on s-plane are lines of constant ω_d consequently they represent lines of constant Tp. Also vertical lines represent constant values of and are therefore lines of constant Ts.
- Finally, since $\zeta = cos\theta$, radial lines represent lines of constant damping ratio. But %Overshoot depends only on ζ .

$$\%\text{OS} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

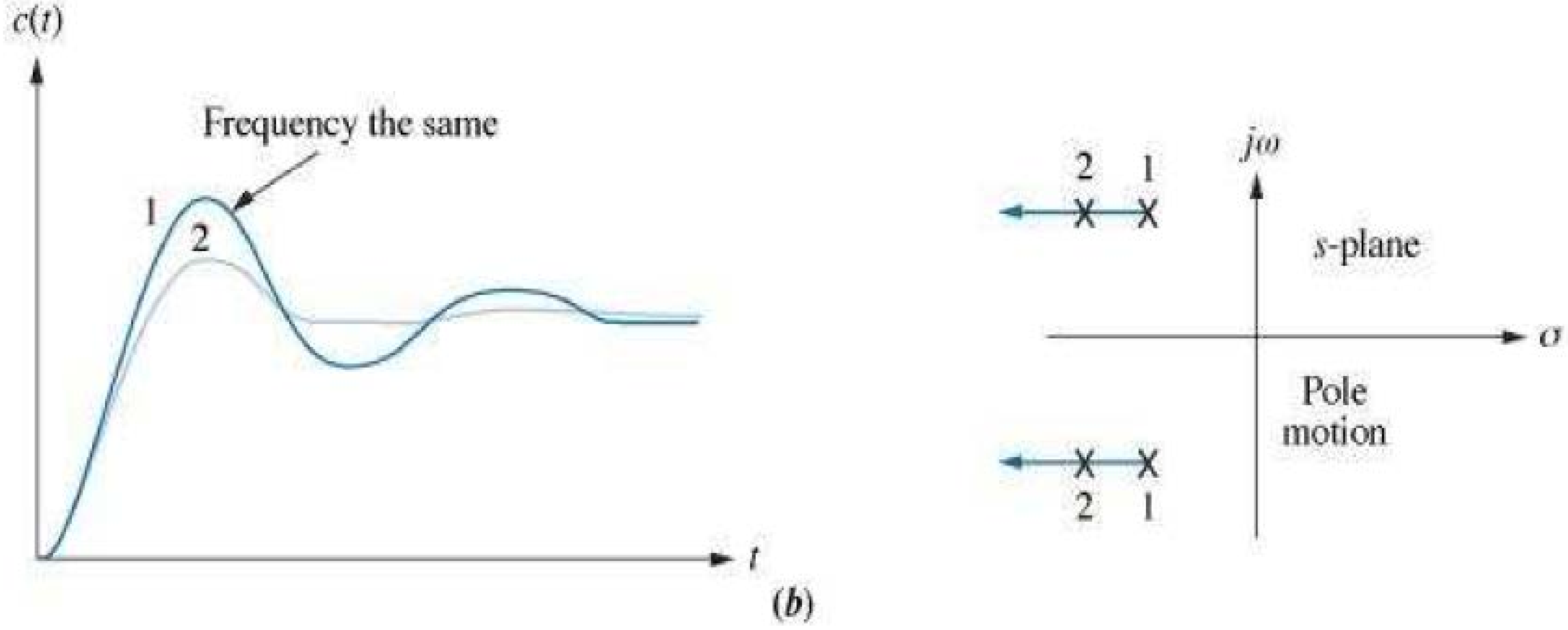


Location Of Poles vs Response (1)

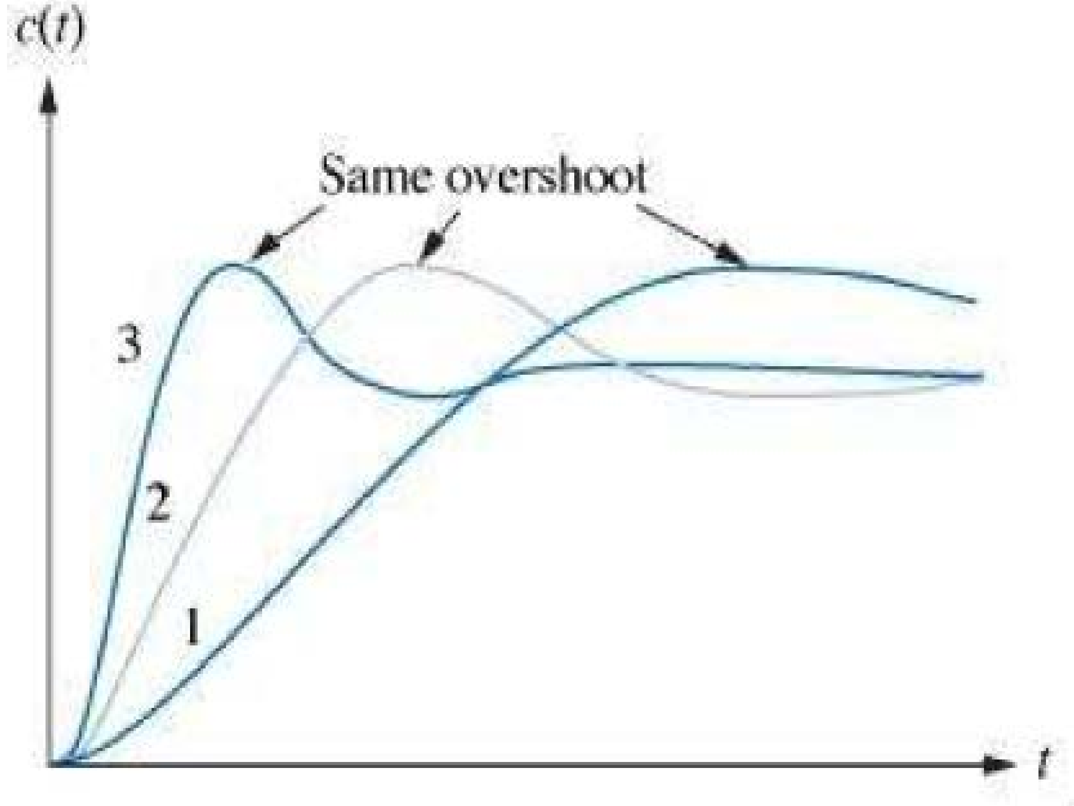




Location Of Poles vs Response (2)



Location Of Poles vs Response (3)



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