Real Time Systems and Control Applications

Contents

Location of Poles

Root Locus

Final Value Theorem

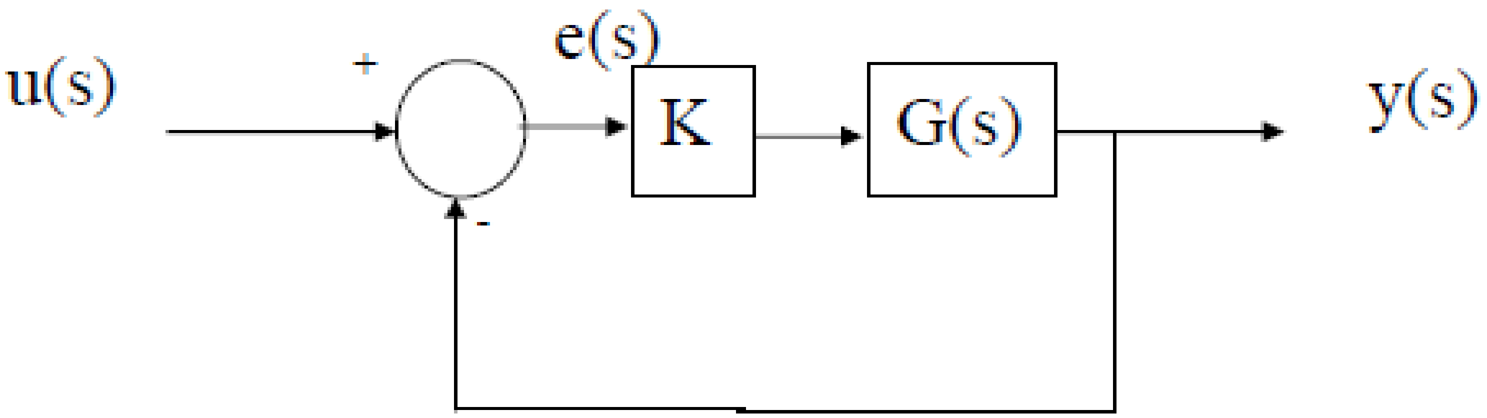
- If a system is stable and has a final constant value, then this theorem can be used to find the steady state value without solving for the system's entire response.

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

- Whether a given function has a final value or not depends on the locations of the poles of its transfer function. For example if there are poles in the right hand side of s-plane or if there are pairs of complex conjugate poles on the imaginary axis, the final value does not exist.

Root Locus

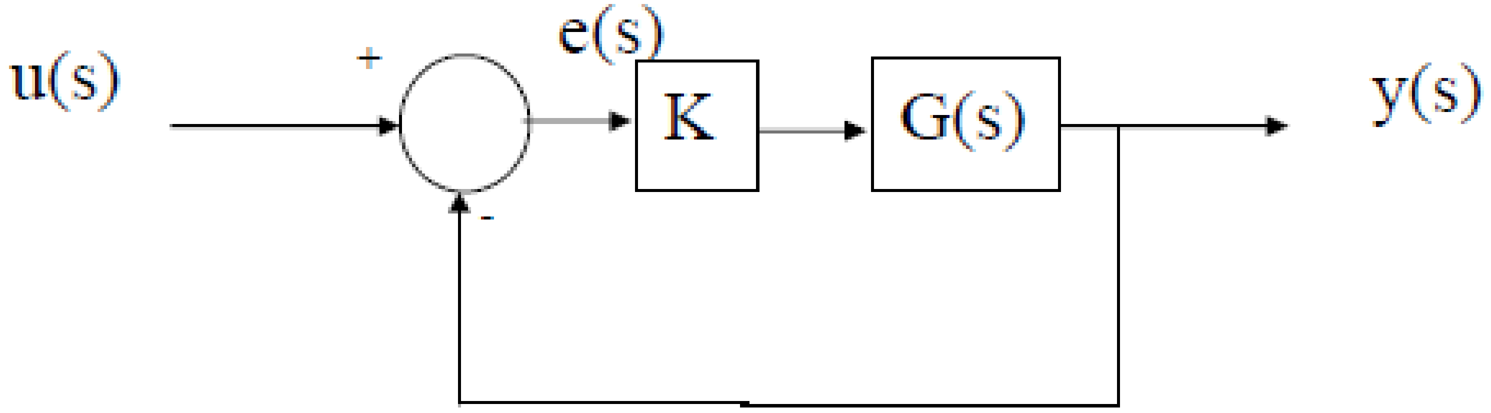
- Root locus is a design and analysis method that **tells us how the roots of a closed loop system change when a parameter such as a gain (K) varies**



Poles of G(s) ➔ Poles of the closed loop system without solving the closed loop transfer function.

Closed Loop System

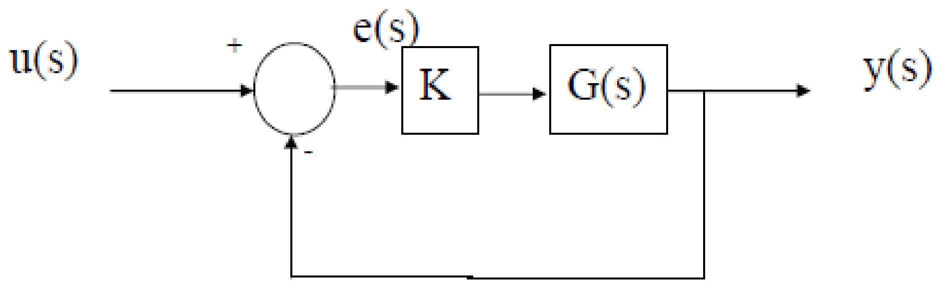
- Consider the following closed loop system:



- The transfer function (TF) of the closed loop system is $\frac{KG(s)}{1+KG(s)}$

Note that the pole locations change as K is varied.

Example



Characteristic Equation: $\mathbf{1 + KG(s) = 0}$

- Consider the transfer function where $G(s) = \frac{1}{s(s+4)}$, The characteristic equation for unity feedback is:

$$1+\frac{K}{s(s+4)} = 0 \text{ or } s^2 + 4s + K = 0$$

- Let us now examine the effect of changing K. The roots can be found by

$$s = \frac{-4 \pm \sqrt{16 - 4K}}{2} = -2 \pm \sqrt{4 - K}$$

Observe The Roots While Increasing K Value

- For K = 0, s = 0 and -4
- For K = 1, s = -0.27 and -3.73
- For K = 2, s = -0.59 and -3.41
- For K = 3, s = -1 and -3
- For K = 4, s = -2 (repeated roots)

- Observe that the two roots start moving closer to each other as K is increased.

- Let us continue…

Continued

- For K = 5, s = $-2 \pm j$
- For K = 8, s = $-2 \pm 2j$
- For K = 13, s =$-2 \pm 3j$
- Observe that for K > 4, the real part of the complex roots remains fixed at -2 while the imaginary part becomes larger and larger in opposite directions towards infinity along a line parallel to the imaginary axis. Since there is no value of K that results in a root lying on the imaginary axis (except origin), nor in the right half of the complex plane, we conclude that the system is stable for all values of gain. If the root locus moves in the right half of the complex plane for certain values of K, we know for which values of gain the system will become unstable.

Some Rules for Sketching Root Locus

- **Number of Branches** is equal to the number of **closed-loop poles**, which equals to the number finite open loop poles or the number of finite open loop zeros.
- **Symmetry**: The root locus is symmetrical about the real axis.
- **Starting and ending points**: The root locus begins at the finite and infinite **poles of open loop transfer function** G(s)H(s) and ends at the finite and infinite **open loop zeros** of G(s)H(s).

Some Rules for Sketching Root Locus

- **Behaviour at infinity**: The root locus approaches straight lines as asymptotes as K approaches infinity. Further the equation of the asymptotes is given by the real axis intercept σ_a and angle θ_a, as follows:

$$\sigma_a = \frac{\sum finite\ poles - \sum finite\ zeros}{\#finite\ poles - \#finite\ zeros}$$

$$\theta_a = \frac{(2k+1)\pi}{\#finite\ poles - \#finite\ zeros}$$

where k = 0, ± 1, ± 2, ± 3, and the angle is given in radians with respect to the positive extension of the real axis.

Note that every function has an equal number of poles and zeros if we include the infinite poles and zeros as well as the finite poles and zeros.

Some Rules for Sketching Root Locus

- **Breakaway or break-in Points**: The breakaway/break-in points are located at the roots of the following equation:

$$\frac{d[G(s)H(s)]}{ds} = 0$$

The gain *K* is minimum/maximum for break-in/breakaway points compared to the gains of all other points belonging to the same real-axis segment of the root locus.

Example 1

Consider the following closed loop system,

where $G(s) = \frac{1}{s(s+2)}$

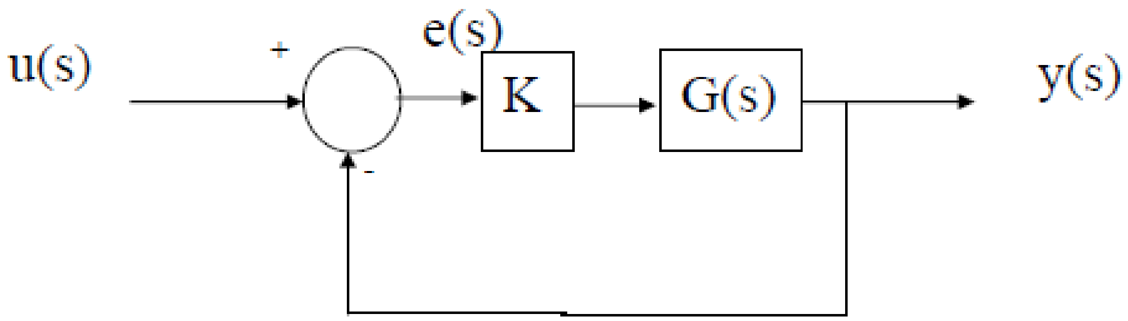
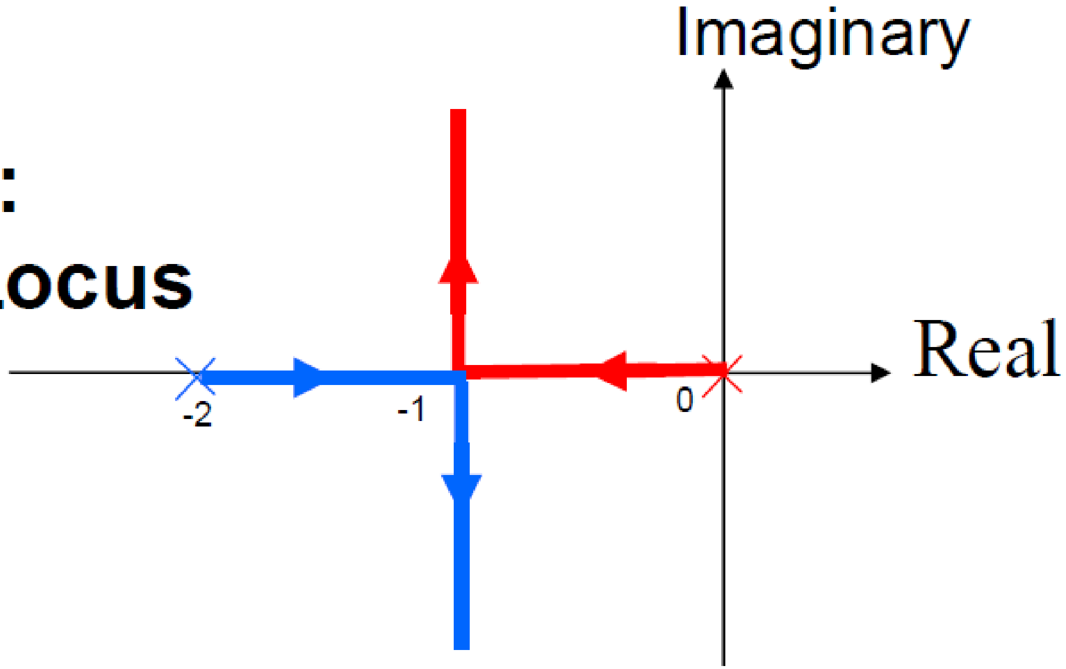
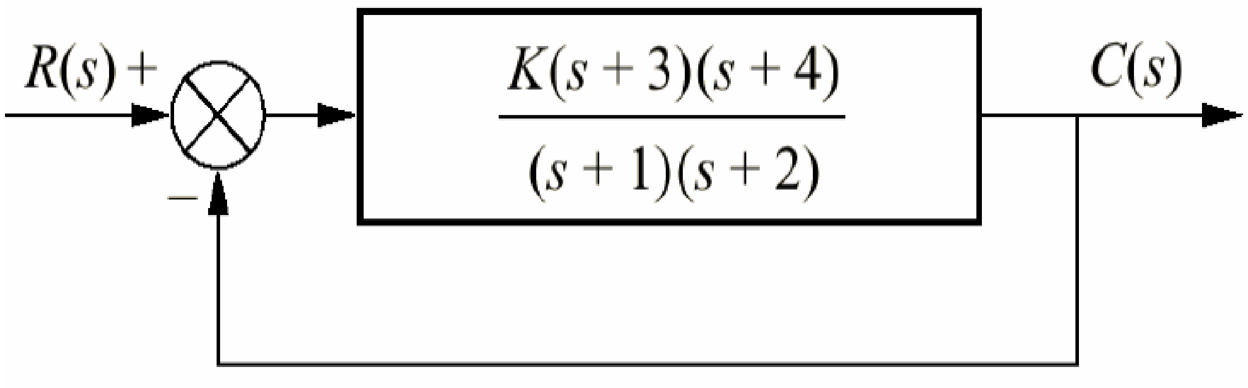


Figure:
Root Locus

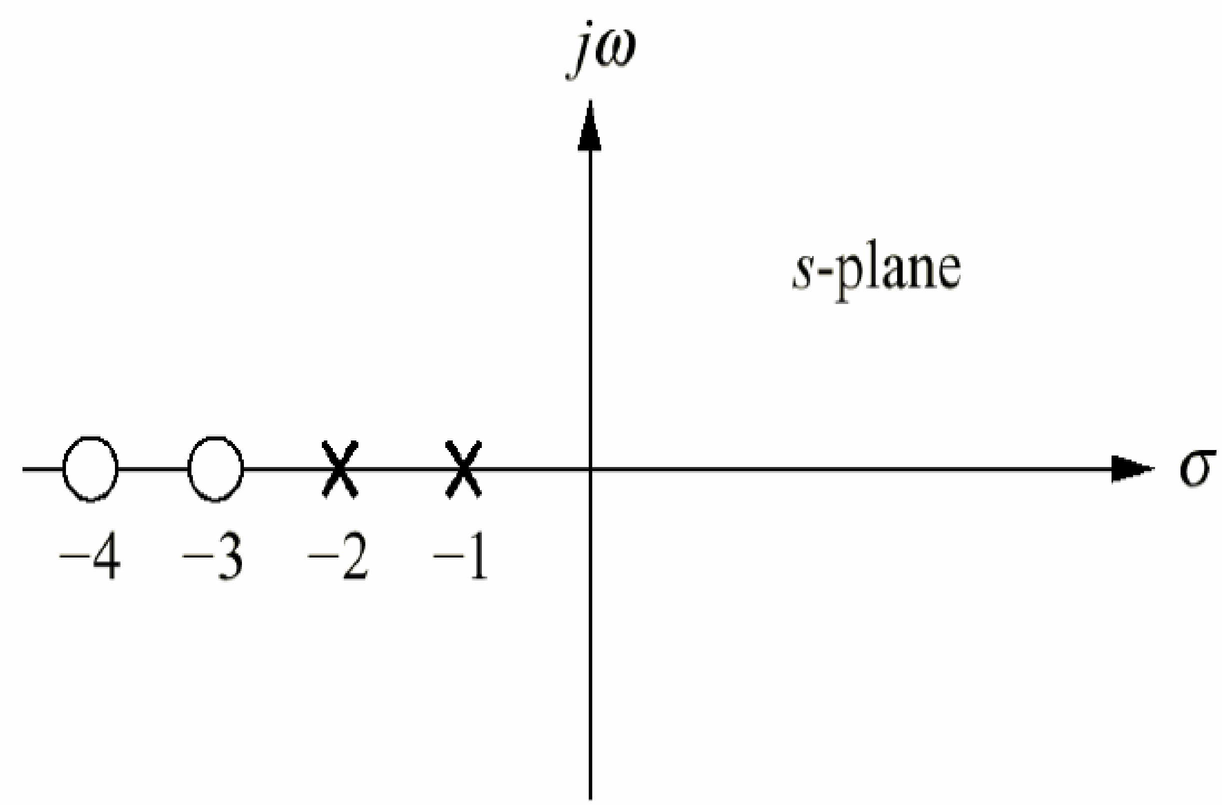


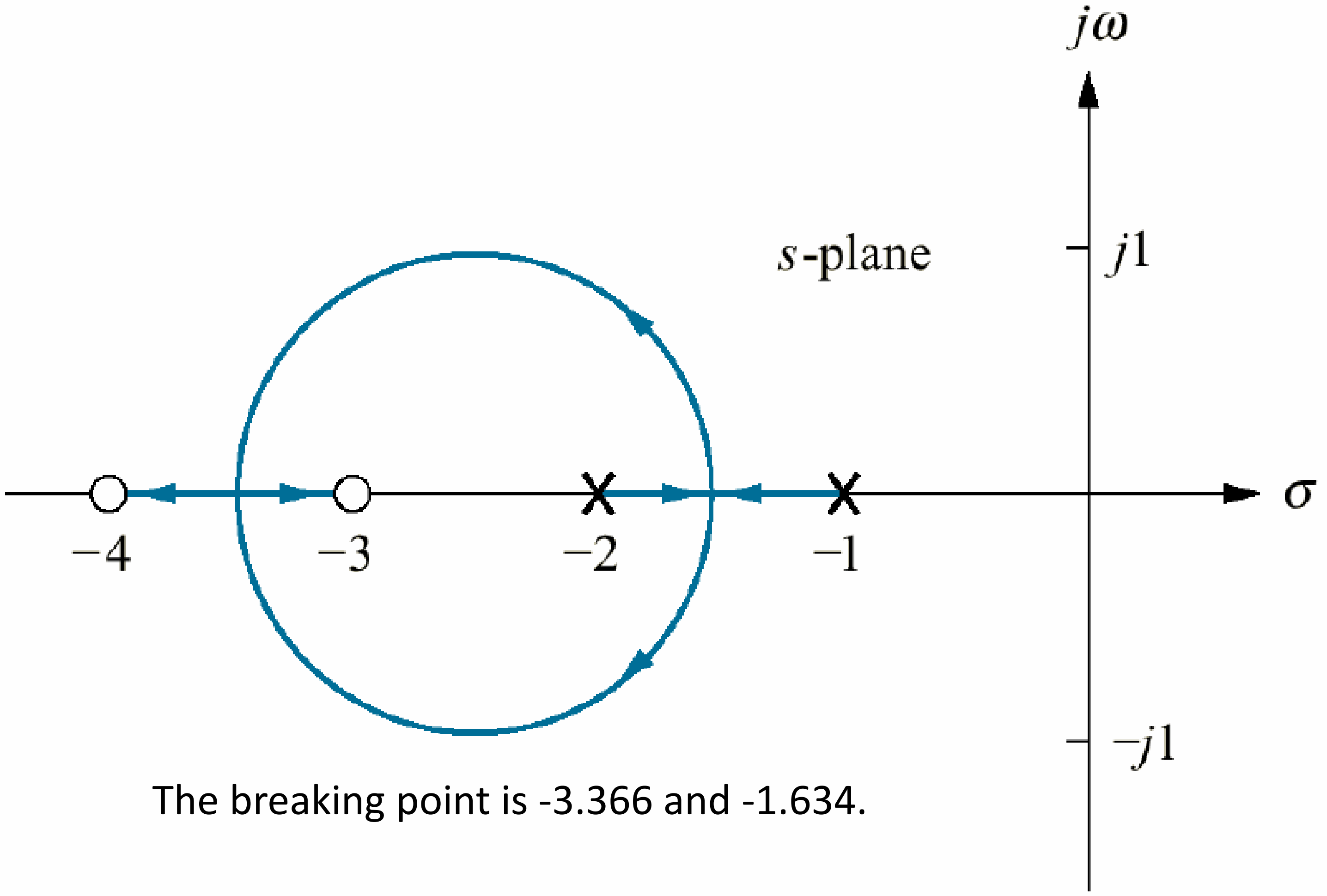
(1) Root locus has 2 branches
(2) Symmetrical about real axis
(3) Start from open-loop poles to open-loop zeros
(4) The break away point is (-1, 0)
(5) $\sigma_a = \frac{-2}{2-0} = -1,\ \theta_a = \frac{\pi}{2}, \frac{3\pi}{2}$

Example 2



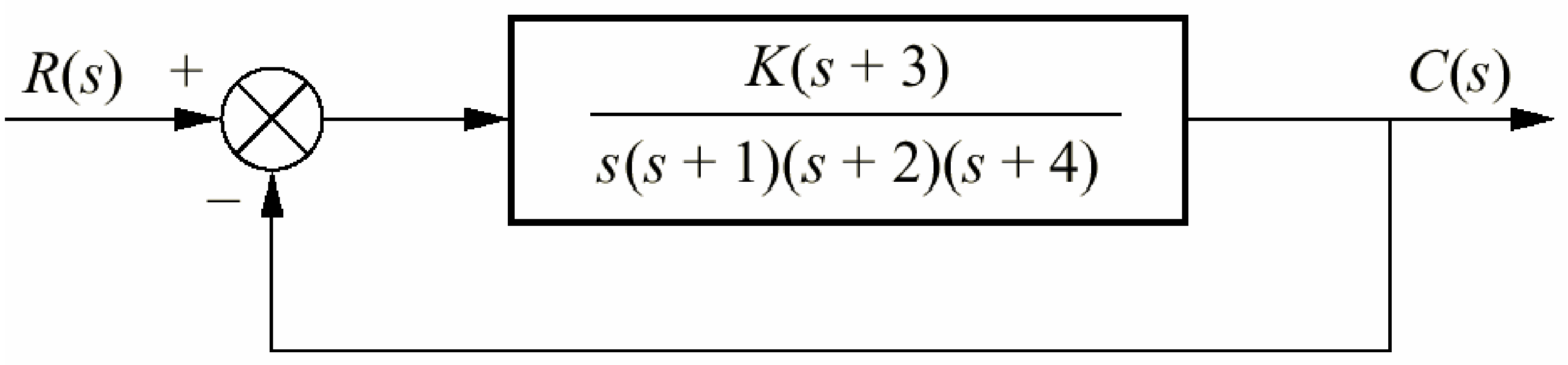
What does the root locus look like if the open-loop zeros and open-loop poles are given?



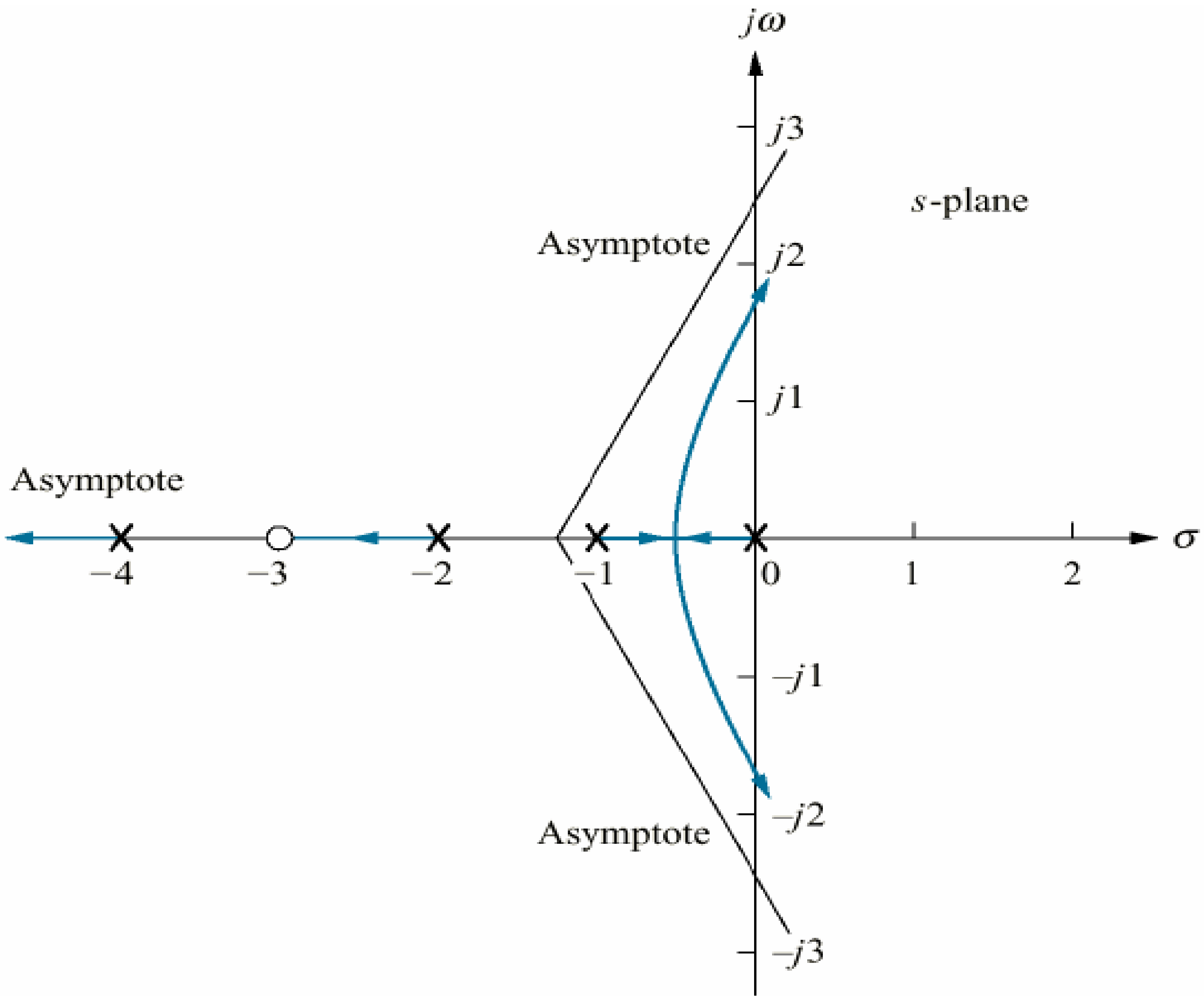


The breaking point is -3.366 and -1.634.

Example 3



- Open-loop poles are 0, -1, -2, and -4.
- Finite open-loop zero is -3, and there are 3 infinite zeros.
- $\#finite\ poles - \#finite\ zeros = 3$
- The asymptotes intercept with real axis at $-\frac{4}{3}$
- The angles of asymptotes are $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$


jω
j3
s-plane
Asymptote
j2
j1
Asymptote
σ
−4
−3
−2
−1
0
1
2
−j1
−j2
Asymptote
−j3

Question:

- For a unity feedback system with $G(s) = 10 / s^2$, what is locus of the closed loop poles?

Note that there are two open loop poles (two zero poles),

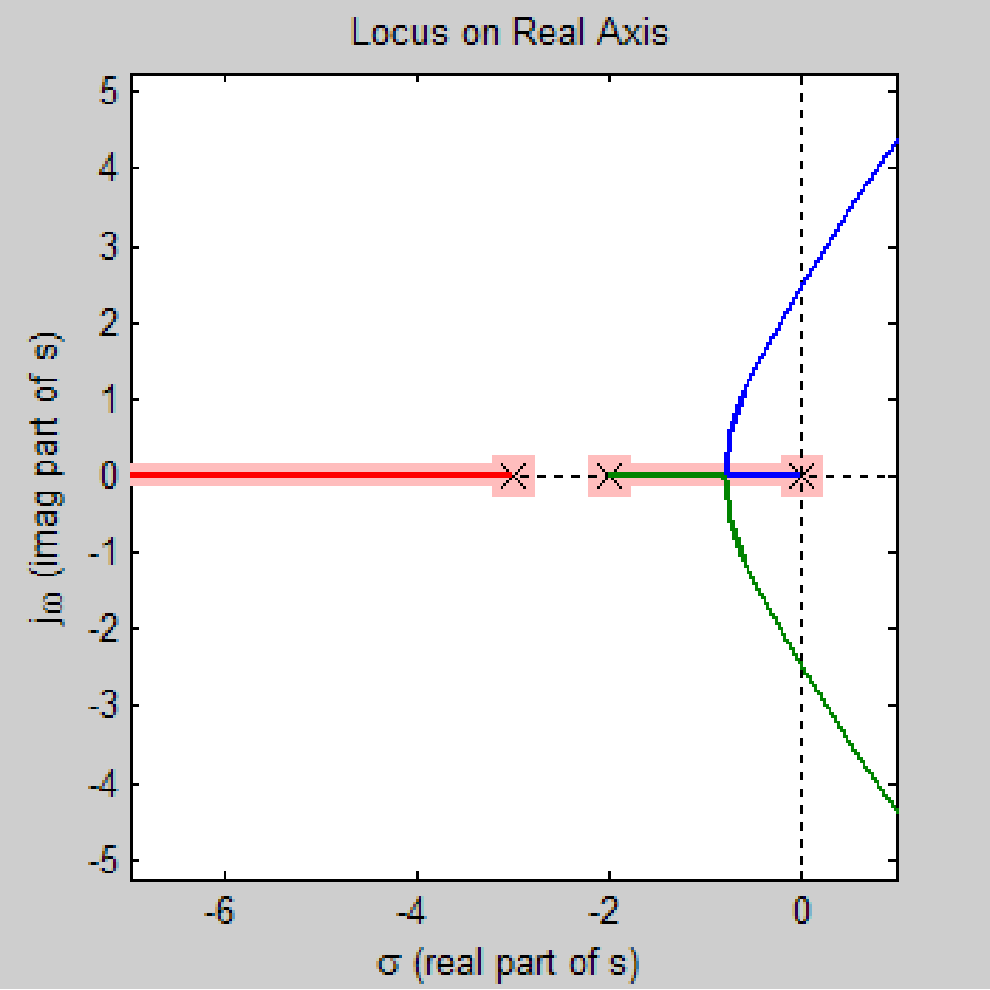
$\sigma_a = 0$, and $\theta_a = \frac{\pi}{2}, \frac{3\pi}{2}$

Question

- If the system is specified by open loop transfer function $G(s)H(s) = K/ s(s+3) (s + 2)$, how many root locus branches proceed to end at infinity? What do they look like?

$d\{1/[s(s+3)(s + 2)]\}/ds=0$

$d[s(s+3)(s + 2)]/ds=0$



- Determine how many branches?

(how many branches go to infinity)

- What do those asymptotes look like?

- Breakaway point?

  A point satisfies $3\ s^2 + 10\ s + 6 = 0$

  s=-2.5 or s=-0.78 (which one?)

More Practice on Root Locus

- Open-loop transfer functions

(1) $\frac{1}{s(s+4)}$

(2) $\frac{s+3}{s^2-s-2}$

(3) $\frac{s+1}{s^3+4s^2+6s+4}$