Real Time Systems and Control Applications

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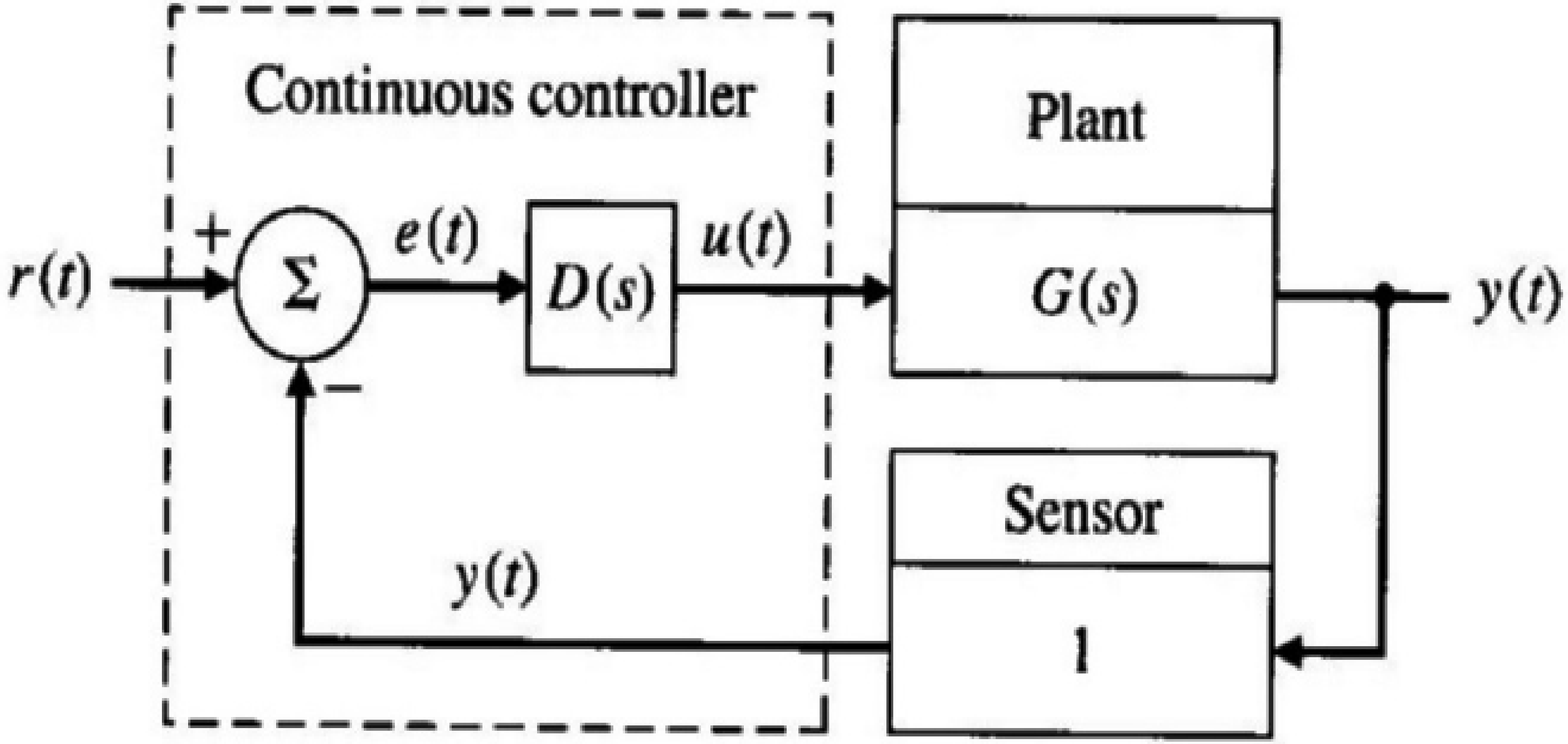
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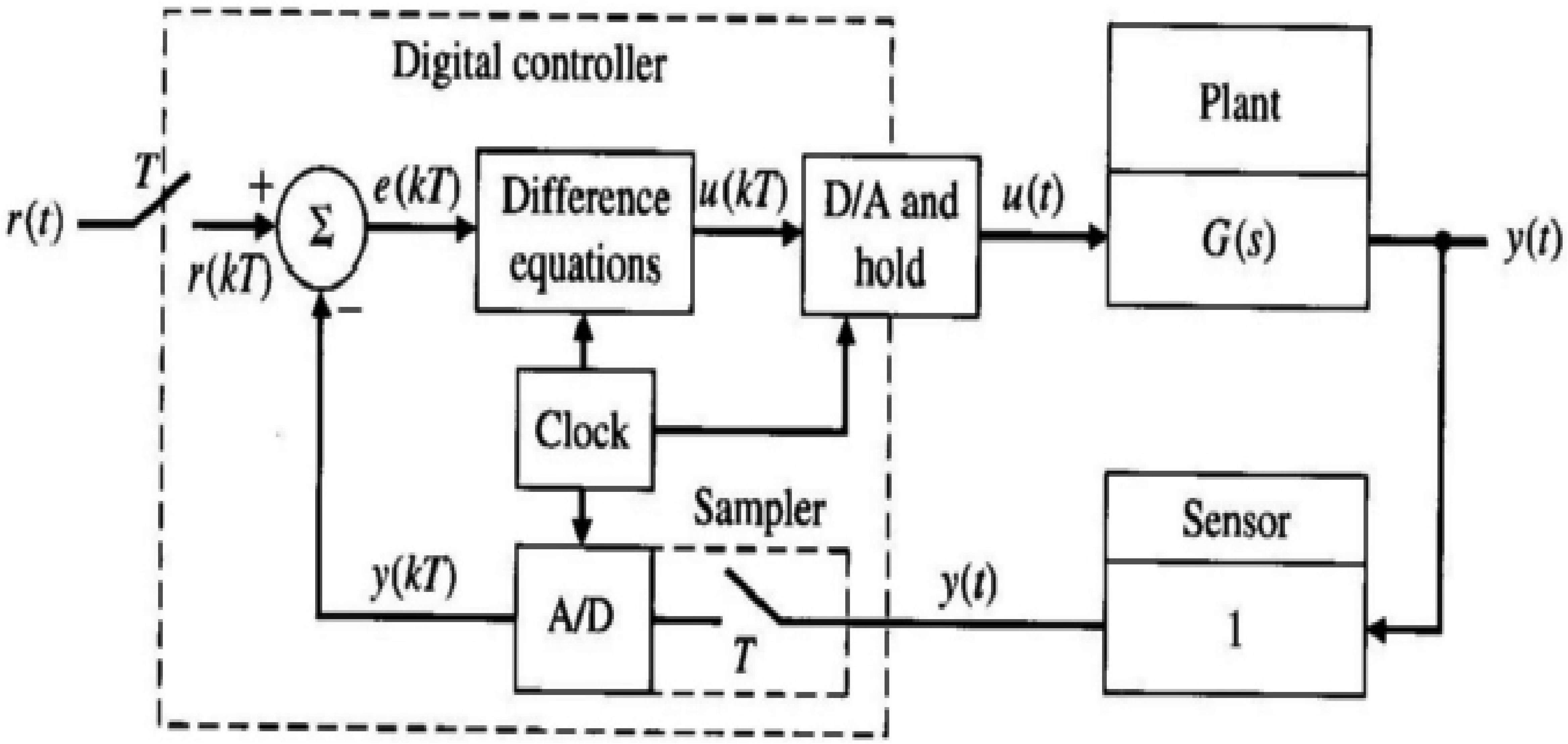
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Goals

- Introduce suitable mathematical models for analysis on discrete systems, so that we can build on the existing knowledge of continuous control systems (CCS).

- Show how we convert CCS transfer functions into equivalent transfer functions for DCS.

- If a stable CCS is transformed into a DCS, determine if the DCS remains stable?

Sampling (Analog to Digital Converter)

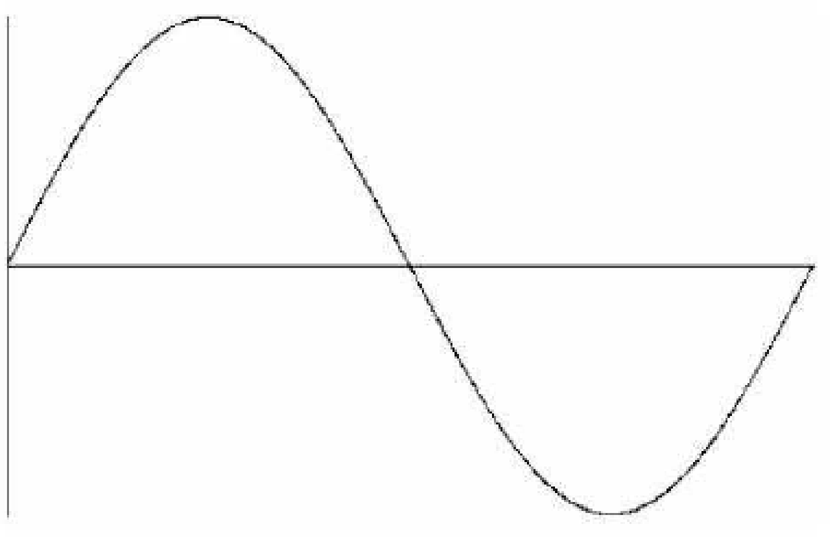
- The analog signal is thus represented by a sequence of sampled values.

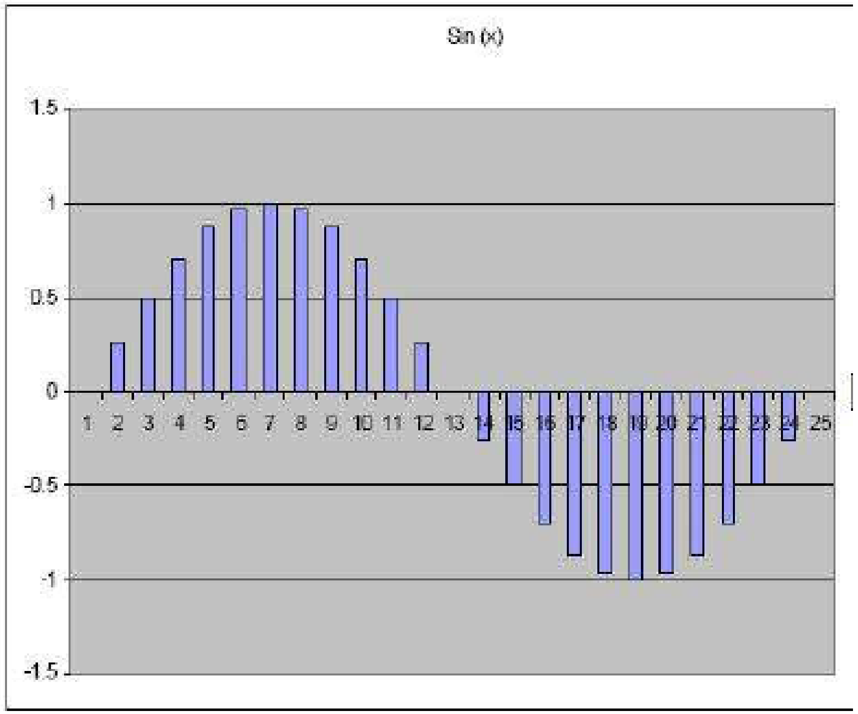
- The conversion of analog signal takes place repetitively at instants of time that are T seconds apart, T is called the sampling period, and 1/T is the sampling rate in cycles per second.

- The accuracy of the digital signal depends on the number of bits used to represent the samples.

Analog to Digital Converter

- Sampling: the process of taking values at discrete time intervals
- Amplitude values are represented using binary numbers, which have finite resolution



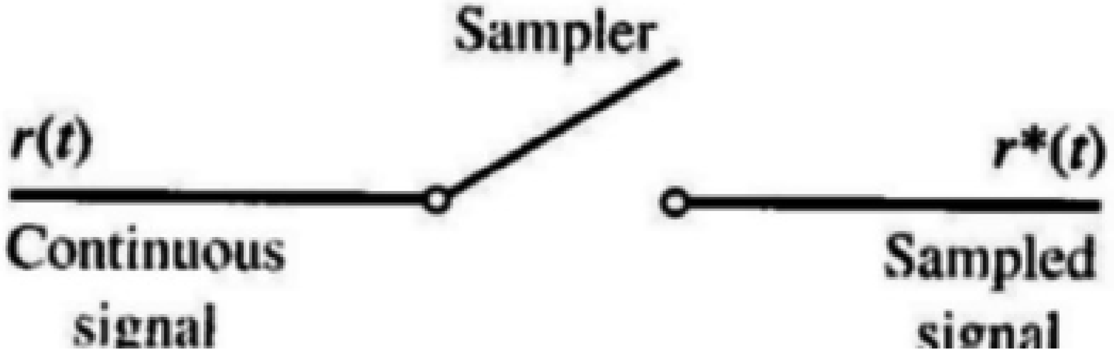


Quantization Error

- An example: Assume that three bits are used to represent a sampled value as a binary number. Let M be the maximum analog voltage that is divided into 8 levels of M/8 volts.
- A 3-bit number can represent all of the eight levels. Values that fall in between these levels must be approximated to next higher/lower binary values.
- This may result in a maximum error of M/16 volts in a sampled digital output. In general **quantization error** is equal to $\frac{1}{2} \times \frac{M}{2^n} = \frac{M}{2^{n+1}}$, where n is the number of bits used for digitization.
- The **resolution** of A/D converter is the minimum value of the output that can be represented as a binary number, i.e., $\frac{M}{2^n}$.

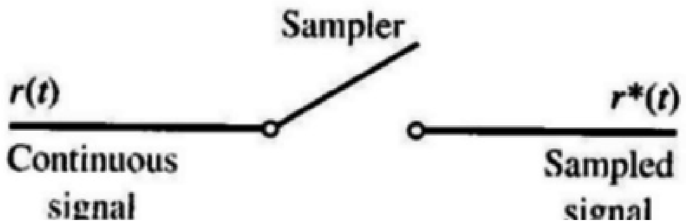
Sampled Data System

- Input signals are available only at sample intervals of time (A2D conversion)
- Thus the reference input r is a sequence of sample values r(kT), instead of r(t)
- A sampler is basically a switch that closes every T seconds for one instant



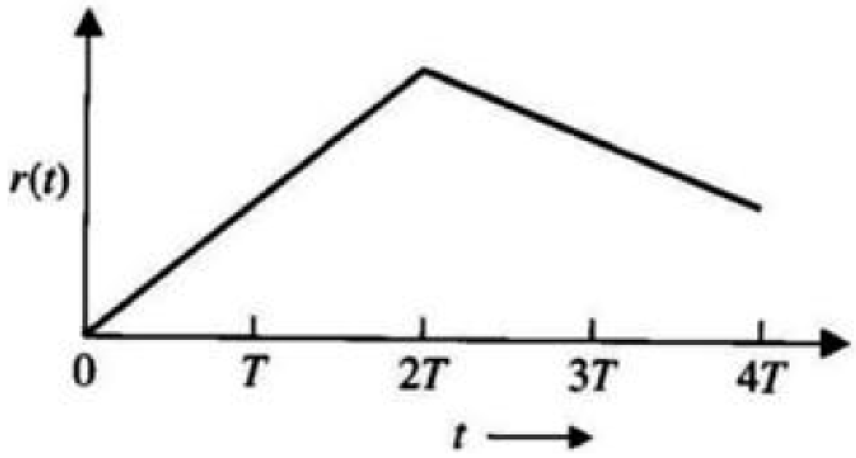
$$r(kT)\,\delta(t-kT)$$

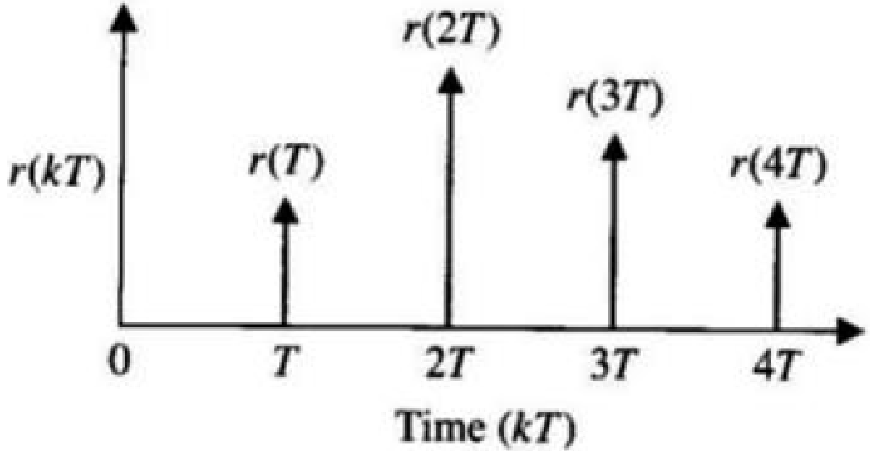
Example of Sampled Data System



$r(kT)\,\delta(t-kT)$

- Assume that we sample a signal r(t) and obtain r*(t).





- We can write

$$r^*(t) = \sum_{k=0}^{\infty} r(kT)\,\delta(t-kT) \quad (t > 0)$$

Transfer Function of Sampled Data System

- Consider $r^*(t) = \sum_{k=0}^{\infty} r(kT)\,\delta(t-kT)$, using Laplace Transform we have:

$$R^*(s) = \mathcal{L}\big(r^*(t)\big) = \sum_{k=0}^{\infty} r(kT)\,e^{-ksT}$$

- We can transform the above series to a more manageable expression

by defining **z-Transform**.

Z-Transform (Laplace Transform of Sampled Data)

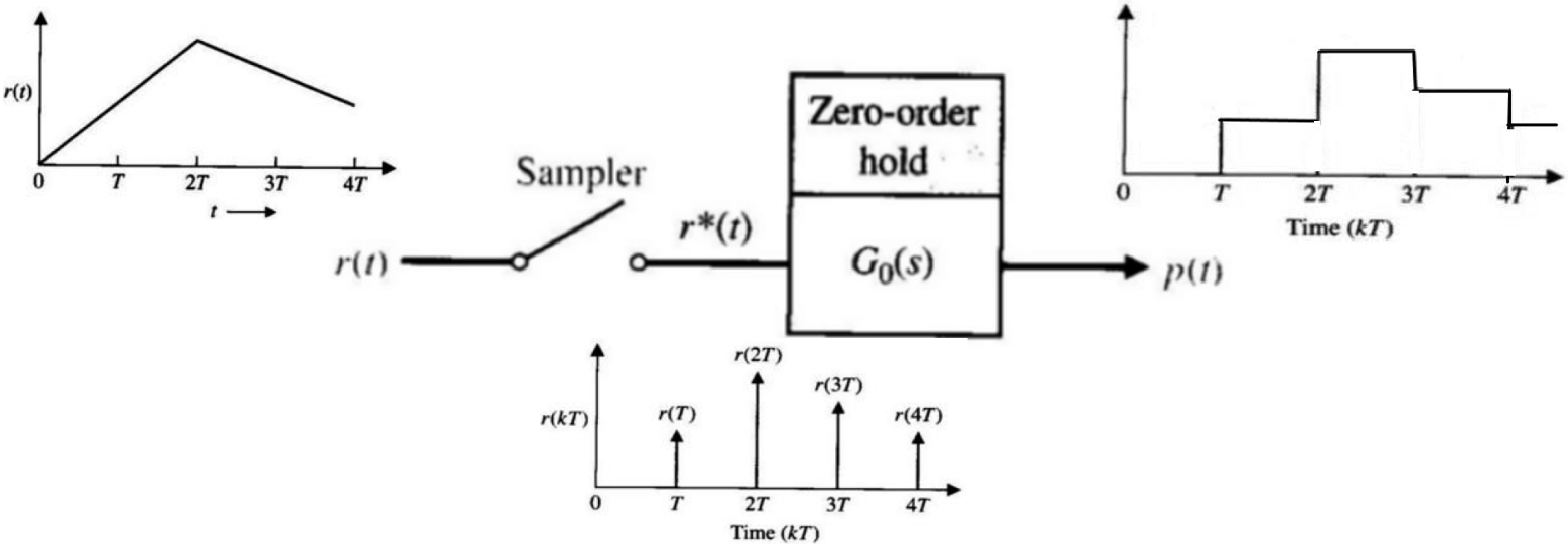
- In $\mathcal{L}\left(r^*(t)\right) = \sum_{k=0}^{\infty} r(kT)\, e^{-ksT}$, if we define $z = e^{sT}$, now the z-Transform is defined as:

$$Z\{r(t)\} = Z\left(r^*(t)\right) = \sum_{k=0}^{\infty} r(kT)\, z^{-k}$$

- In general, the z-Transform of a function $f(t)$ is defined as:

$$\boldsymbol{Z\{f(t)\} = F(z) = \sum_{k=0}^{\infty} f(kT)\, z^{-k}}$$

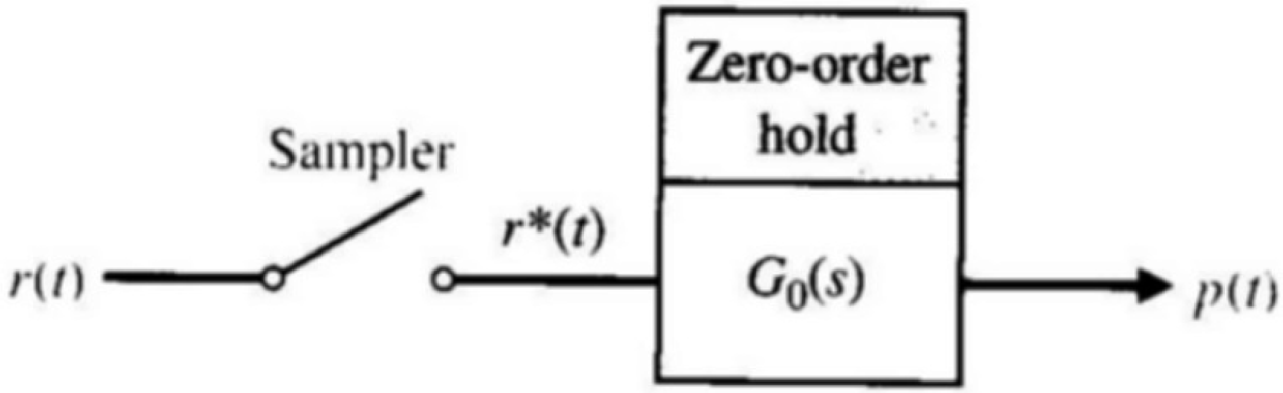
A2D and D2A

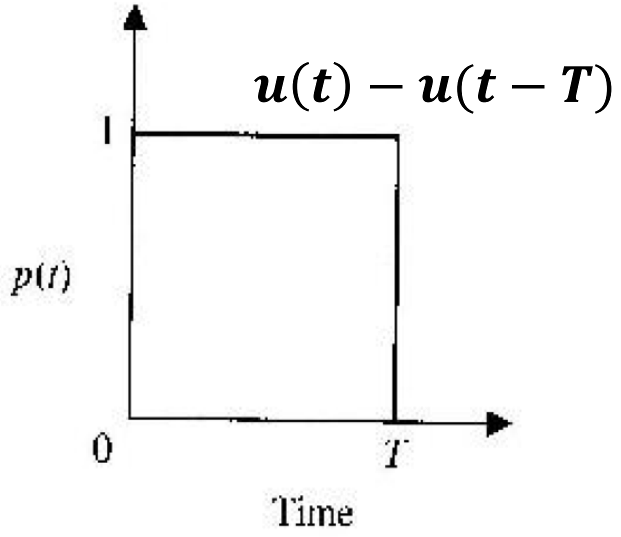


- r(t) and p(t) are not same!
- If T is small enough, p(t) is close to r(t).

Zero Order Hold (Digital to Analog Converter)

- A device that holds the sampled signal r(t) to a constant value for duration of the sampling period. (i.e., The zoh takes the value r(kT) and holds it constant for kT < t < (k+1)T.)





Transfer Function of Zero Order Hold $\mathcal{L}\big(u(t) - u(t - T)\big) = \frac{1}{s} - \frac{e^{sT}}{s}$

Example 1

- Let $f(t) = e^t$ for t ≥ 0, what is $Z\big(f(t)\big)$?

$$Z\big(f(t)\big) = \sum_{k=0}^{\infty} e^{kT} z^{-k} = \sum_{k=0}^{\infty} (\frac{e^T}{z})^k = \frac{1}{1 - \frac{e^T}{z}} = \frac{z}{z - e^T}$$

Sum of geometric series

We also have $Z(e^{at}) = \frac{z}{z - e^{aT}}$

Example 2

$$Z(e^{at}) = \frac{z}{z - e^{aT}}$$

- Let $f(t) = \sin(\omega t)$ for t ≥ 0. What is $Z\big(f(t)\big)$?

- We write $\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$ by Euler's Formula.

$$Z(f(t)) = Z(\frac{e^{j\omega t}}{2j} - \frac{e^{-j\omega t}}{2j}) = \frac{1}{2j}(\frac{z}{z - e^{j\omega T}} - \frac{z}{z - e^{-j\omega T}})$$

$$= \frac{1}{2j}\ \frac{z(e^{j\omega T} - e^{-j\omega T})}{z^2 - z(e^{j\omega T} + e^{-j\omega T}) + 1} = \frac{z\sin(\omega T)}{z^2 - 2z\ \cos(\omega T) + 1}$$

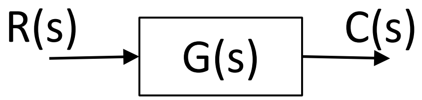
Partial Table of z-Transforms

TABLE 2-3 z-Transforms

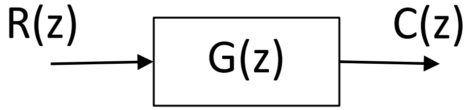
Sequence	Transform
$\delta(k-n)$	z^{-n}
1	$\frac{z}{z-1}$
k	$\frac{z}{(z-1)^2}$
k^2	$\frac{z(z+1)}{(z-1)^3}$
a^k	$\frac{z}{z-a}$
ka^k	$\frac{az}{(z-a)^2}$
$\sin ak$	$\frac{z \sin a}{z^2 - 2z \cos a + 1}$
$\cos ak$	$\frac{z(z - \cos a)}{z^2 - 2z \cos a + 1}$
$a^k \sin bk$	$\frac{az \sin b}{z^2 - 2az \cos b + a^2}$
$a^k \cos bk$	$\frac{z^2 - az \cos b}{z^2 - 2az \cos b + a^2}$

Transfer Function

- Transfer function of a continuous system is $G(s) = \frac{C(s)}{R(s)}$, where $\mathrm{R(s)} = \mathcal{L}(r(t))$ and $\mathrm{C(s)} = \mathcal{L}(c(t))$.



- The transfer function of a sampled system with discrete input and output is given by $G(z) = \frac{C(z)}{R(z)}$, where $\mathrm{R}(z)$ and $\mathrm{C}(z)$ are z-transform of sampled input and output.



Finding the Discrete Transfer Function G(z)

Steps:

(1) Start with the frequency domain transfer function G(s).

(2) Find g(t) using inverse Laplace Transform Tables.

(3) Then use z-transform tables to find G(z).

Note that the discrete transfer function depends on sampling period T.

Example

- Find the transfer function in z-domain of $G(s) = \frac{s^2+4s+3}{s^3+6s^2+8s}$

$$G(s) = \frac{s^2+4s+3}{s^3+6s^2+8s} = \frac{0.375}{s} + \frac{0.25}{s+2} + \frac{0.375}{s+4}$$

$$\mathrm{G}(t) = \mathcal{L}^{-1}\left[\frac{0.375}{s} + \frac{0.25}{s+2} + \frac{0.375}{s+4}\right] = 0.375 + 0.25e^{-2t} + 0.375e^{-4t}$$

$$G(z) = Z\left(0.375 + 0.25e^{-2t} + 0.375e^{-4t}\right) = 0.375\frac{z}{z-1} + 0.25\frac{z}{z-e^{-2T}} + 0.375\frac{z}{z-e^{-4T}}$$

For a given sampling period T, (i.e., T=0.1), we have

$$G(z) = \frac{z^3 - 1.658z^2 + 0.68z}{z^3 - 2.489z^2 + 2.038z - 0.5488}$$

Inverse z-Transform

- $G(z) \to x(k)$

- Methods:

(1) Power Series Method

(2) Partial-Fraction Expression Method

(3) Inversion-Formula Method

Power Series Method

- If $G(z)$ is expressed as the ratio of two polynomials in z, we can write $G(z)$ in a power series:

$$G(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots$$

- Example:

$$G(z) = \frac{z}{z^2 - 3z + 2}$$

Using long division, we have $G(z) = z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} + \cdots$

Hence, $g(0) = 0, g(T) = 1, g(2T) = 3, g(3T) = 7$ …

Partial-Fraction Expression Method

- Let's consider

$$\frac{z}{z-a} = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + a^4z^{-4} + \cdots$$

- Write $G(z)$ in partial fraction expression, and use the above equation to find $g(k)$.

- Example: $G(z) = \frac{z}{(z-1)(z-2)} = \frac{-z}{z-1} + \frac{z}{z-2} = \sum_{k=0}^{\infty}(-1 + 2^k)z^{-k}$
- Therefore, $g(kT) = 2^k - 1$

How to get partial fraction expression of $G(z) = \frac{z}{z^2-3z+2}$?

- $G(z) = \frac{z}{z^2-3z+2} \Rightarrow G(z) = \frac{z}{(z-1)(z-2)}$
- Write $G(z) = \frac{A}{(z-1)} + \frac{B}{(z-2)} = \frac{z}{(z-1)(z-2)}$, so we can solve A and B.

$$\frac{z}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)} = \frac{(A+B)z-(2A+B)}{(z-1)(z-2)}$$

So $(A+B) = 1$ and $(2A+B) = 0$

We got A=-1 and B=2

- We know $G(z) = \frac{-1}{(z-1)} + \frac{2}{(z-2)}$

Another Form of Z Transfer Function

- $G(z) = \frac{z}{(z-1)(z-2)}$
- We first get the partial fraction expression of $\frac{G(z)}{z} = \frac{1}{(z-1)(z-2)}$
- We get $\frac{G(z)}{z} = \frac{-1}{z-1} + \frac{1}{z-2}$ ➔ $G(z) = \frac{-z}{z-1} + \frac{z}{z-2}$

On the previous page, we got $G(z) = \frac{-1}{(z-1)} + \frac{2}{(z-2)}$.

Are these G(z) we got both correct?

Are they the same?

$$\frac{z}{z-2} = 1 + 2z^{-1} + 2^2 z^{-2} + 2^3 z^{-3} + \cdots = \sum_{k=0}^{\infty} 2^k z^{-k}$$

$$\frac{z}{z-1} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots = \sum_{k=0}^{\infty} z^{-k}$$

- $G(z) = \frac{-1}{(z-1)} + \frac{2}{(z-2)} = \left[\frac{-z}{(z-1)} + \frac{2z}{(z-2)}\right] z^{-1}$

$$g(kT) = -1 + 2 \times 2^{k-1} = 2^k - 1$$

- $(2)\ G(z) = \frac{-z}{z-1} + \frac{z}{z-2}$

$$g(kT) = 2^k - 1$$