Real Time Systems and Control Applications

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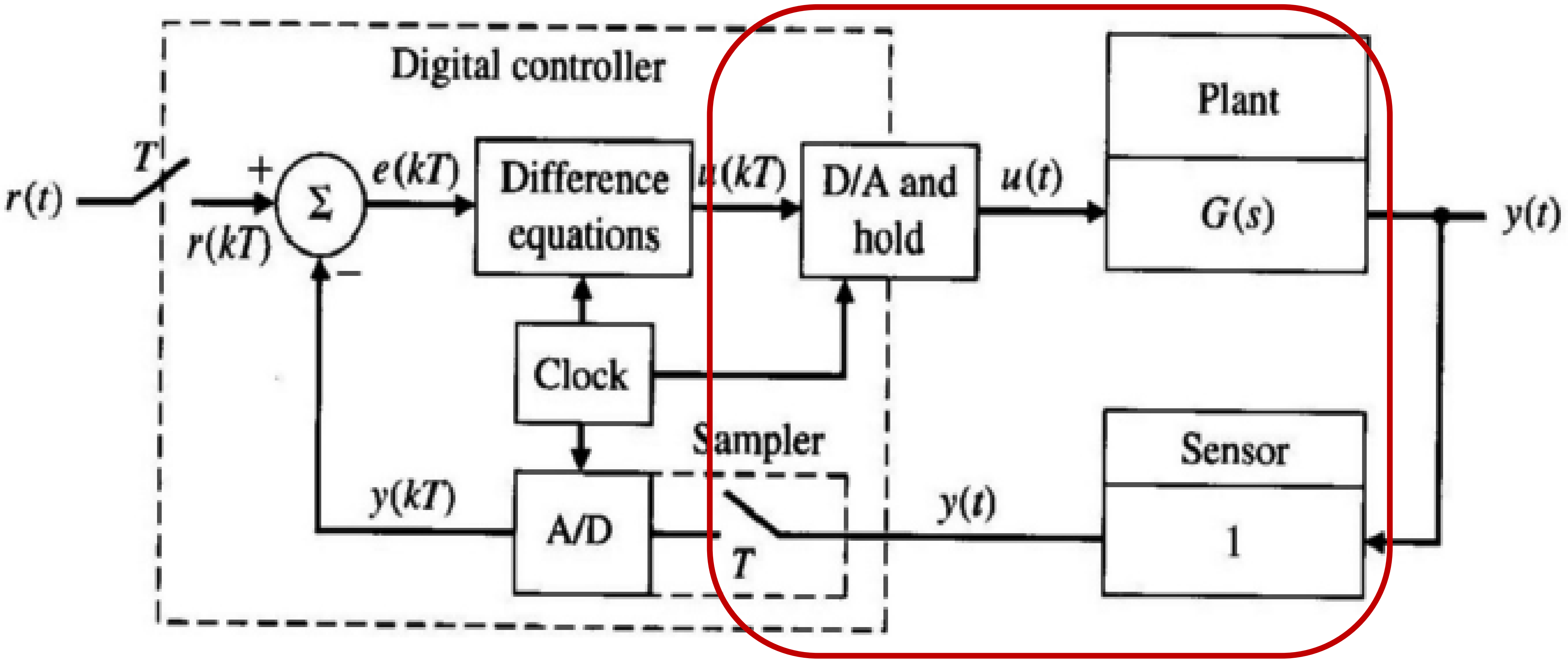
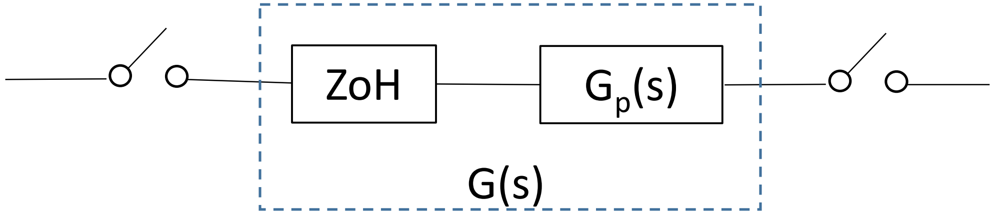
What is the TF of the subsystem in the box



Given $G_p(s)$, Find $G(s)$ and $G(z)$



Transfer Function of Zero Order Hold

$$\mathcal{L}(\boldsymbol{u(t)}-\boldsymbol{u(t-T)}) = \frac{1}{s} - \frac{e^{-sT}}{s}$$

- In frequency domain $G(s) = ZoH(s) \cdot G_p(s)$
- We know that $ZoH(s) = \frac{1-e^{-Ts}}{s}$
- So $G(s) = \left[\frac{1-e^{-Ts}}{s}\right] G_p(s) = \frac{G_p(s)}{s} - e^{-Ts}\frac{G_p(s)}{s}$
- $G(z) = Z[\frac{G_p(s)}{s}] - Z[e^{-Ts}\frac{G_p(s)}{s}] = Z[\frac{G_p(s)}{s}] - z^{-1}Z[\frac{G_p(s)}{s}]$

$$G(z) = (1-z^{-1})Z[\frac{G_p(s)}{s}]$$

Find $G(z)$ if $G_p(s) = \frac{s+2}{s+1}$.

$$G(z) = (1 - z^{-1})Z[\frac{G_p\,(s)}{s}]$$

$$G^*(s) = \frac{G_p\,(s)}{s} = \frac{s+2}{s(s+1)} = \frac{2}{s} - \frac{1}{s+1}$$

Taking Inverse Laplace Transform:

$$g^*(t) = 2 - e^{-t}$$

Hence,

$$g^*(kT) = 2 - e^{-kT}$$

$$G^*(z) = \frac{2z}{z-1} - \frac{z}{z-e^{-T}}$$

We then have

$$G(z) = \mathrm{Z}\big(G^*(s)\big) - z^{-1}\mathrm{Z}\big(G^*(s)\big) = G^*(z) - z^{-1}G^*(z)$$

$$G^*(z) = \frac{2z}{z-1} - \frac{z}{z-e^{-T}}$$

Determine $G(z)$ when T=0.5 sec

- Substituting T=0.5, we have

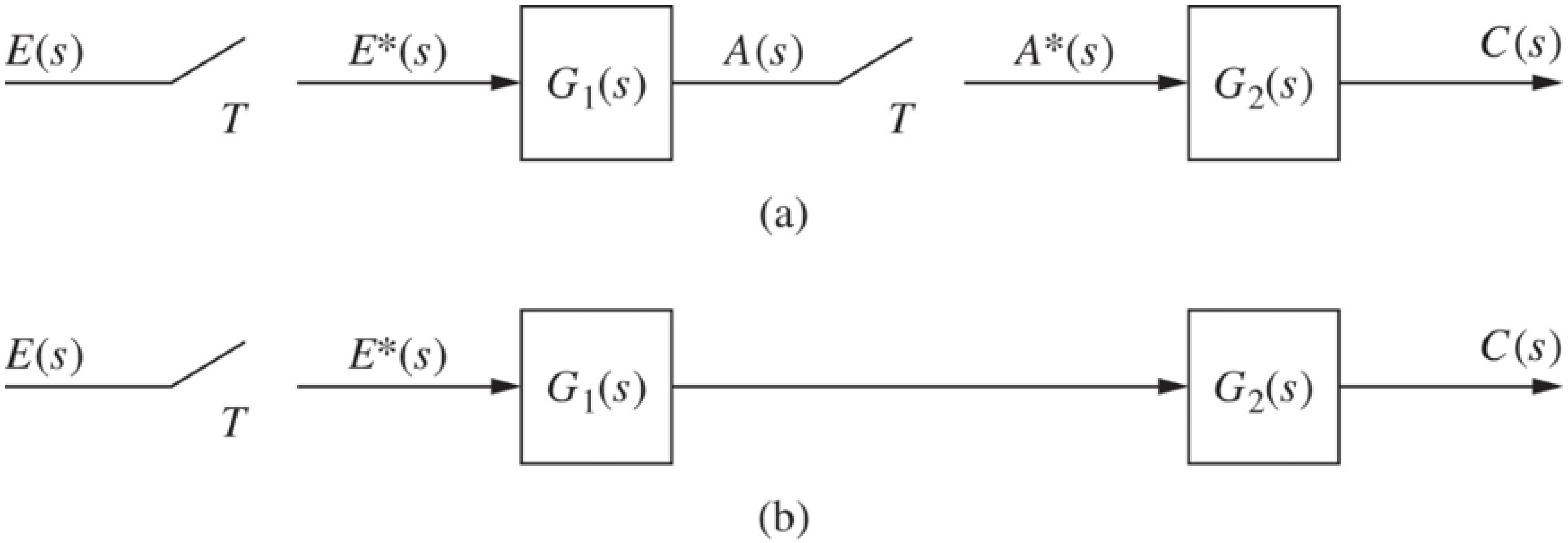
$$G^*(z) = \frac{2z}{z-1} - \frac{z}{z-e^{-T}} = \frac{2z}{z-1} - \frac{z}{z-0.607}$$

Hence, $G(z) = \left(1 - z^{-1}\right)G^*(z) = \frac{z-1}{z}\left[\frac{2z}{z-1} - \frac{z}{z-0.607}\right]$

$$G(z) = \frac{z-0.213}{z-0.607}$$

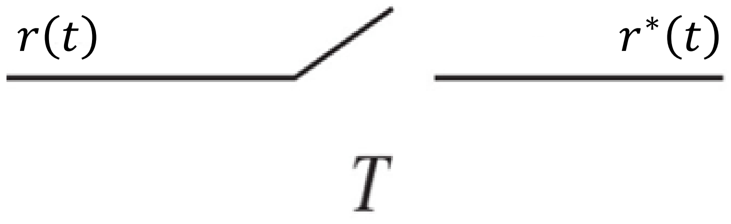
Block Diagram Reduction

Are these sampled-data system the same?



- For (a), we have $C(z) = G_1(z)\ G_2(z)E(z)$
- For (b), we have $C(z) = Z[G_1(s)\ G_2(s)]\ E(z)$

Recall the presence of sampler

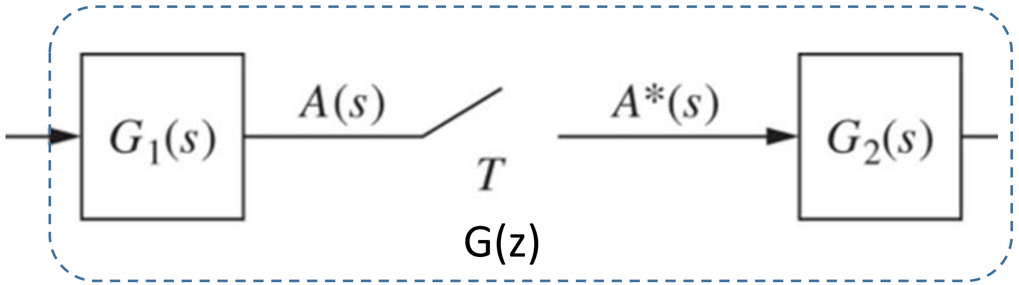


$$r^*(t) = r(0)\delta(t) + r(T)\delta(t-T) + r(2T)\delta(t-2T) + \cdots$$

Hence, $R^*(s) = \sum_{n=0}^{\infty} r(nT)\, e^{-nTs}$.

Consider $R(s) = \int_0^{\infty} r(t)\; e^{-st}\, dt$

$R(s) \neq \mathrm{R}^*(s)$, but $R(z) = \mathrm{R}^*(z)$

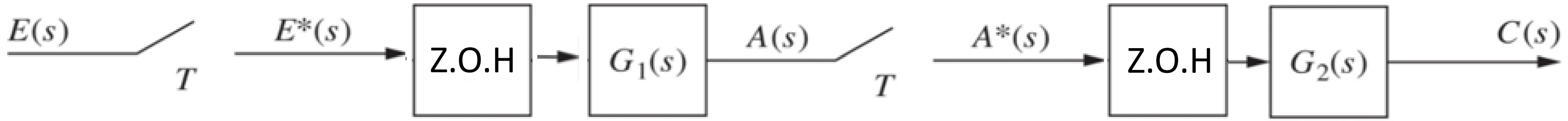


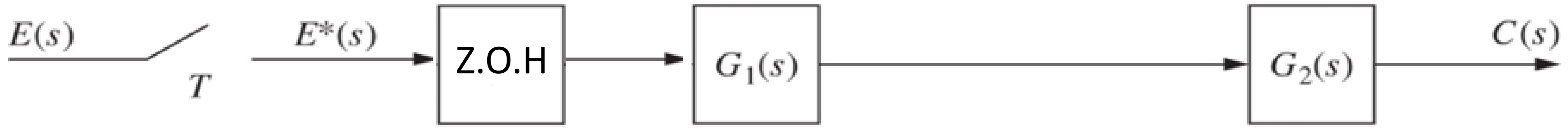
$\mathrm{G}(z) = \mathrm{Z}(G_1(s))\mathrm{Z}(G_2(s)) = G_1(z)\, G_2(z)$ √

$\mathrm{G}(z) = \mathrm{Z}(G_1(s)G_2(s))$ ×

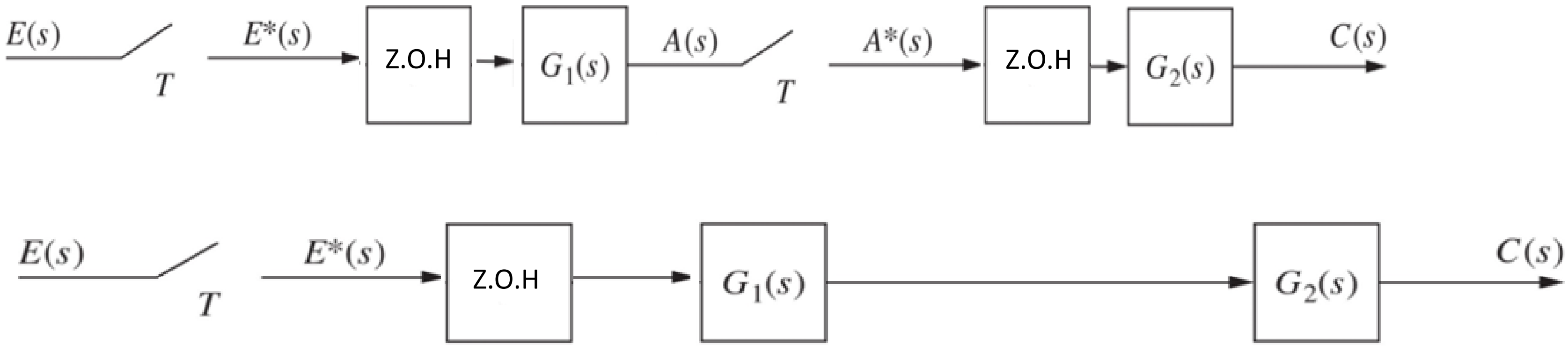
Example

- Given $G_1(s) = \frac{1}{s}$ and $G_2(s) = \frac{1}{s+1}$, respectively. Determine the discrete transfer function G(z) of the open loop sampled data system in the following to cases.





Example



- For (a):

$$G(z) = Z\big(\mathrm{ZoH}(s)\ G_1(s)\big)Z\big(\mathrm{ZoH}(s)\ G_2(s)\big)$$

$$G_a(z) = \left[\frac{z-1}{z}\ Z(\frac{1}{s^2})\right]\left[\frac{z-1}{z}\ Z(\frac{1}{s(s+1)})\right] = [\frac{T}{z-1}][\frac{1-e^{-T}}{z-e^{-T}}]$$

- For (b):

The product of $G_1(s)\ G_2(s)$ must be evaluated before taking the z-transform.

$$G_b(z) = Z\big(\mathrm{ZoH}(s)\ G_1(s)G_2(s)\big) = \frac{z-1}{z}\ Z\left(\frac{1}{s^2(s+1)}\right) = \frac{z-1}{z}\ Z(\frac{1}{s^2} + \frac{1}{(s+1)} - \frac{1}{s})$$

$$G_b(z) = \frac{z-1}{z}[\frac{Tz}{(z-1)^2} + \frac{z}{z-e^{-T}} - \frac{z}{z-1})] = \frac{T}{z-1} + \frac{z-1}{z-e^{-T}} - 1$$

Continued…

- Based on previous page

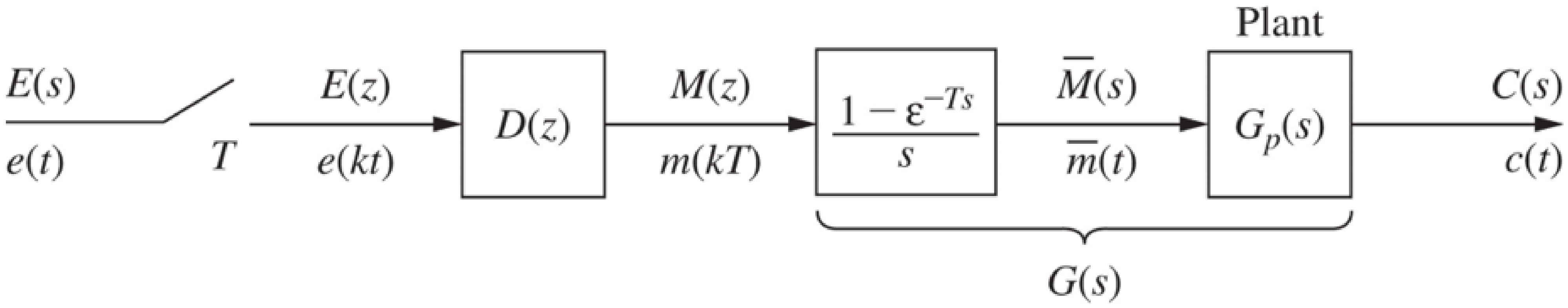
$$G_a(z) = \frac{T - Te^{-T}}{(z-1)(z-e^{-T})}$$

and

$$G_b(z) = \frac{T}{z-1} + \frac{z-1}{z-e^{-T}} - 1 = \frac{Tz - Te^{-T} + (z-1)(e^{-T}-1)}{(z-1)(z-e^{-T})}$$

- They are not the same

Model for the Open-loop System

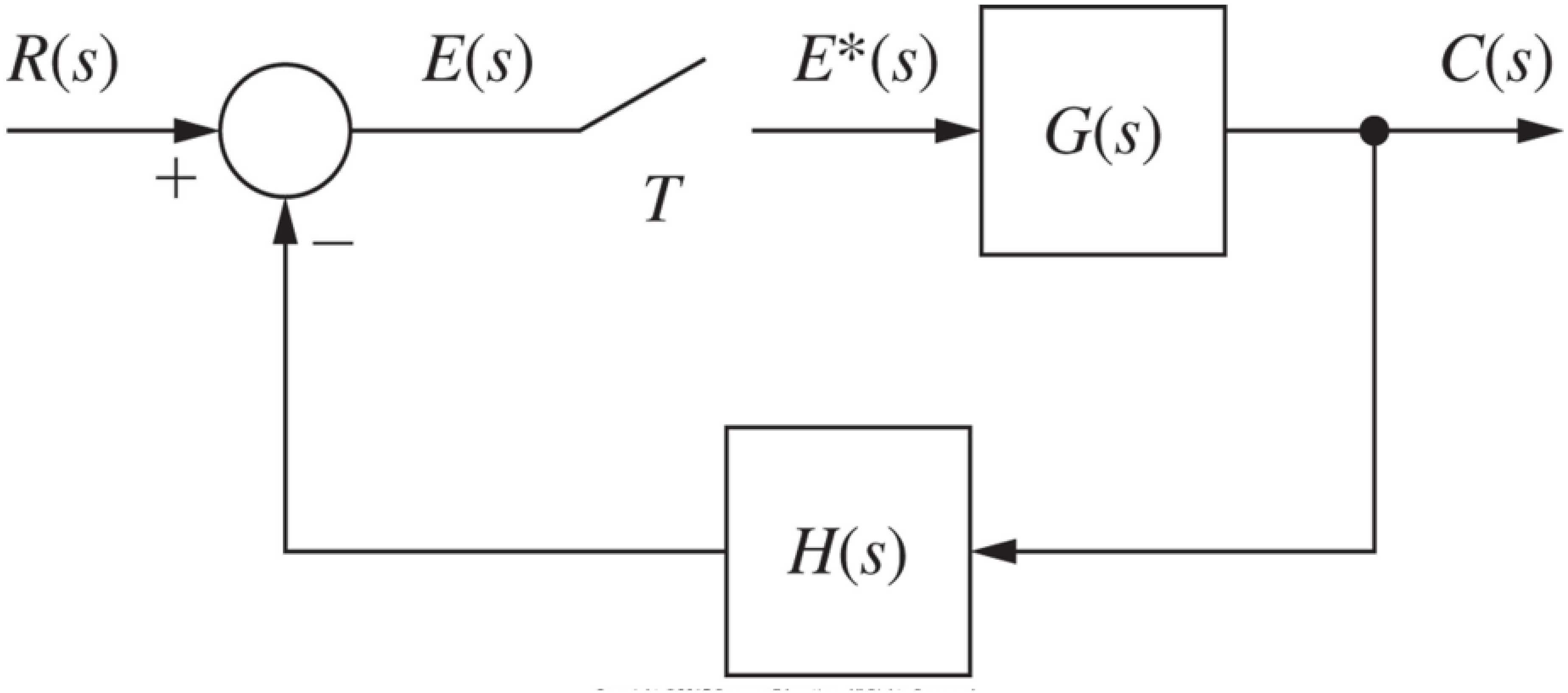


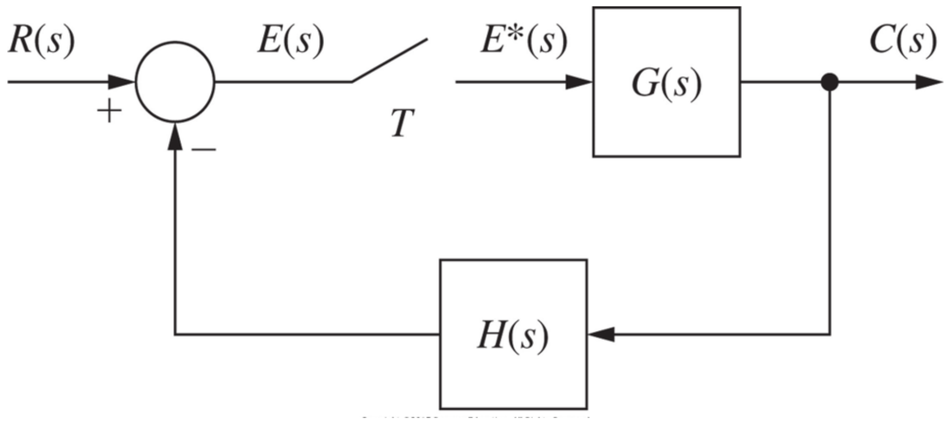
The output of open-loop System is

$$C(z) = G(z)D(z)E(z)$$

Then the $c(kT)$ can be obtained accordingly.

Closed Loop Sample Data System





We have:

$\boldsymbol{C(s) = G(s)\,E^*(s)}$ and $E(s) = R(s) - H(s)C(s)$

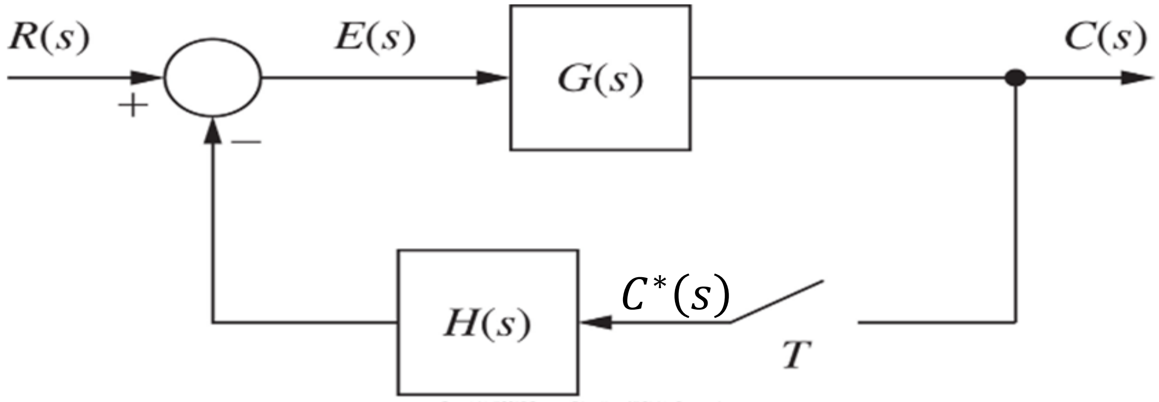
Hence, $\boldsymbol{E(s) = R(s) - G(s)H(s)E^*(s)}$

$$\therefore E(z) = R(z) - Z[G(s)H(s)]E(z)$$

$E(z) = \frac{R(z)}{1+Z[G(s)H(s)]}$ and $C(z) = G(z)\,E(z)$

$$\therefore C(z)/R(z) = \frac{G(z)}{1 + Z[G(s)H(s)]}$$

Closed Loop TF Using Digital Sensing Device



$$C(s) = G(s)[R(s) - H(s)C^*(s)] = G(s)R(s) - G(s)H(s)C^*(s)$$

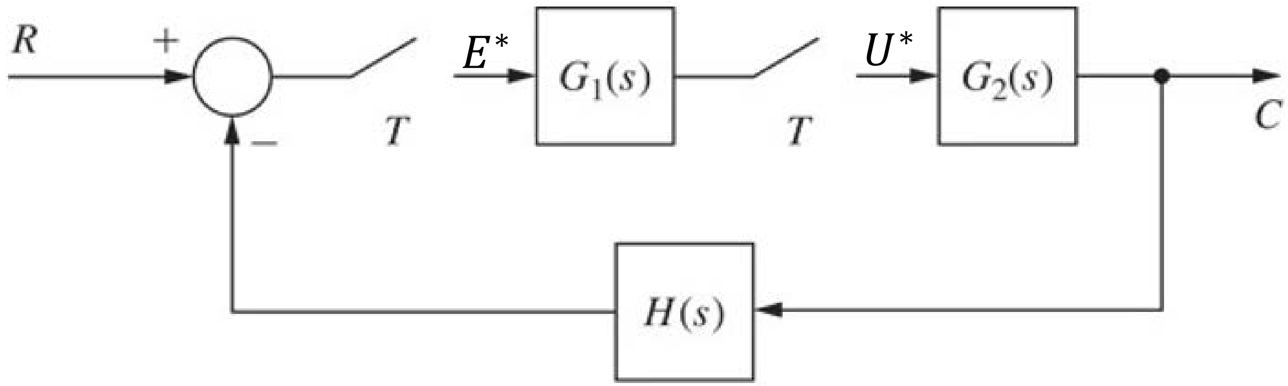
$$C(z)(1 + Z(G(s)H(s))) = Z[G(s)R(s)]$$

$$\therefore C(z) = \frac{Z[G(s)R(s)]}{(1 + Z(G(s)H(s)))}$$

Closed Loop TF Using Digital Controller

$$C(s) = G_2(s)U^*(s)$$
$$U(s) = G_1(s)\,E^*(s)$$
$$E(s) = R(s) - G_2(s)H(s)\,U^*(s)$$

$G_1(s)$: A Controller implemented by Computer

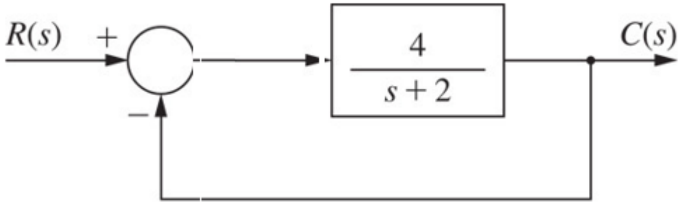


Applying z-transform, we have:

$$C(z) = G_2(z)\,U^*(z)$$
$$U(z) = G_1(z)\,E^*(z)$$
$$E(z) = R(z) - Z[G_2(s)H(s)]\,U^*(z)$$

Hence, $\frac{C(z)}{R(z)} = \frac{G_1(z)G_1(z)}{1+G_1(z)\,Z(G_2(s)H(s))}$

Time Response



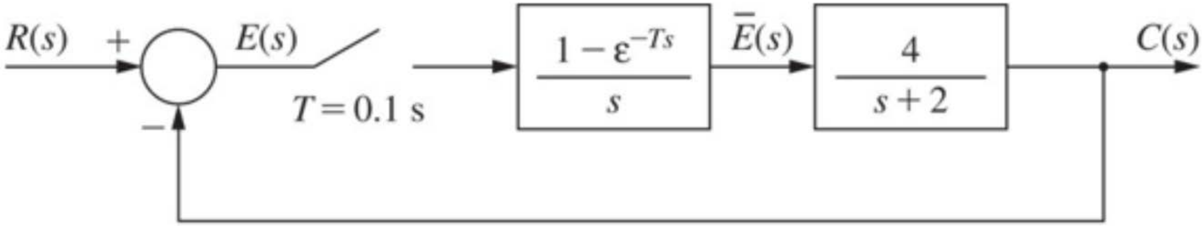
- Consider a continuous system, the closed loop transfer function is

$$T_s = \frac{4}{s+6}$$

Hence the unit step response is

$$c(t) = 0.667(1 - e^{-6t})$$

By adding a sampler and a z.o.h., we get:



Time Response

- We can express the system output as

$$T(z) = \frac{G(z)}{1 + G(z)}$$

where $G(z) = Z\left[\frac{1-e^{-Ts}}{s}\frac{4}{s+2}\right] = \frac{z-1}{z}Z\left[\frac{4}{s(s+2)}\right] = \frac{z-1}{z}\frac{2\left(1-e^{-2T}\right)z}{(z-1)(z-e^{-2T})}$

If $T = 0.1s$, $G(z) = \frac{0.3625}{z-0.8187}$.

Hence,

$$T(z) = \frac{G(z)}{1 + G(z)} = \frac{0.3625}{z - 0.4562}$$

Time Response

- Since $\mathrm{R}(z) = \frac{z}{z-1}$, we have $C(z) = \frac{G(z)}{1+G(z)} R(z)$.

- Hence, $C(z) = \frac{0.3625}{z-0.4562} \frac{z}{z-1} = \frac{0.667z}{z-1} - \frac{0.667z}{z-0.4562}$

- $c(kT) = 0.667[1 - 0.4562^k]$

Time Response