Real Time Systems and Control Applications

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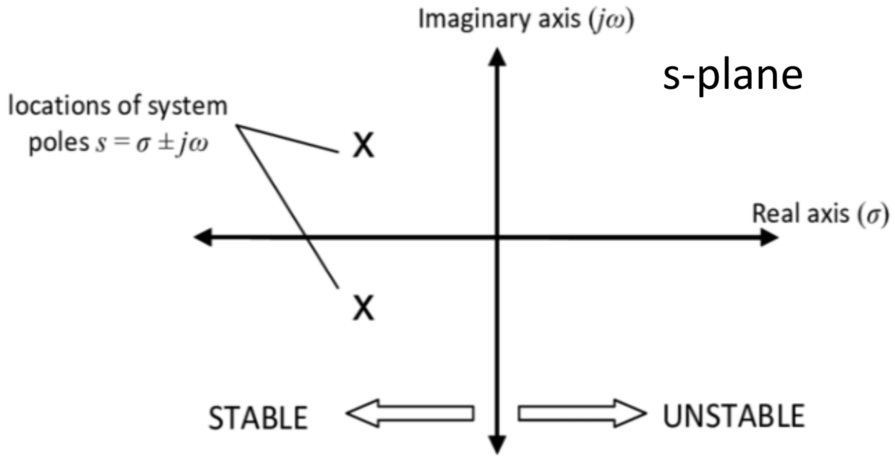
Recall Z-Transform Formula

Laplace Transform of sampled data

$$\mathcal{L}(f^*(t)) = \sum_{k=0}^{\infty} f(kT)\, e^{-ksT}$$

Type equation here.

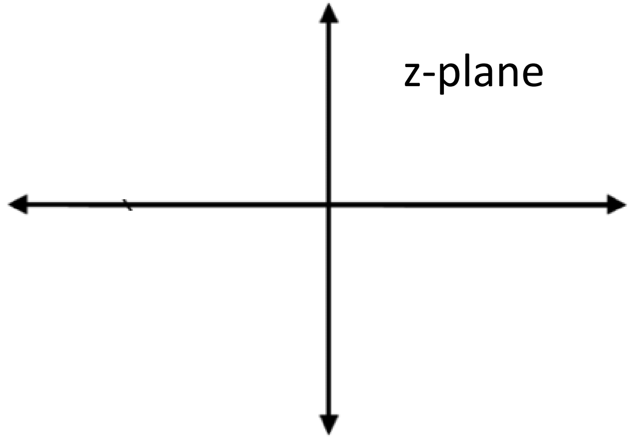
G(s)



Define $z = e^{sT}$, now the z-Transform is defined as:

$$\mathrm{Z}\{f(t)\} = Z(f^*(t)) = \sum_{k=0}^{\infty} f(kT)\, z^{-k}$$

G(z)



Mapping Stability Region from s-plane to z-plane

- From continuous control, we know that: the region of stability is the left half of s-plane. Given G(s) ➔ G(z), determine the stability region on z-plane?

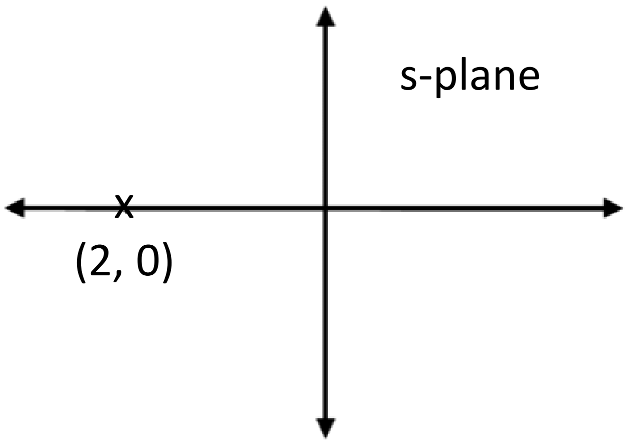
- By definition, $z = e^{sT}$. Let $s = \alpha + j\omega$, then

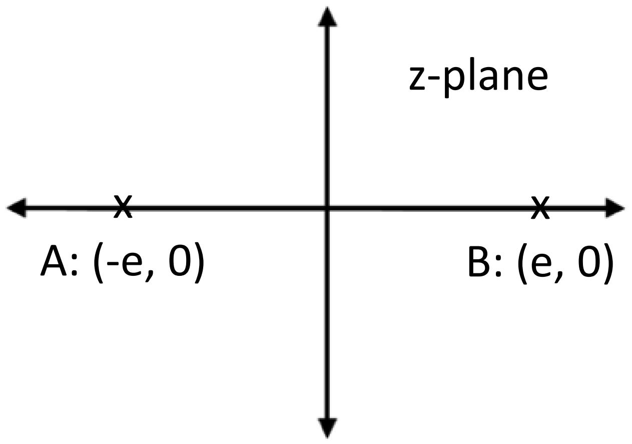
$$z = e^{sT} = e^{T(\alpha + j\omega)} = e^{\alpha T} e^{j\omega T}$$

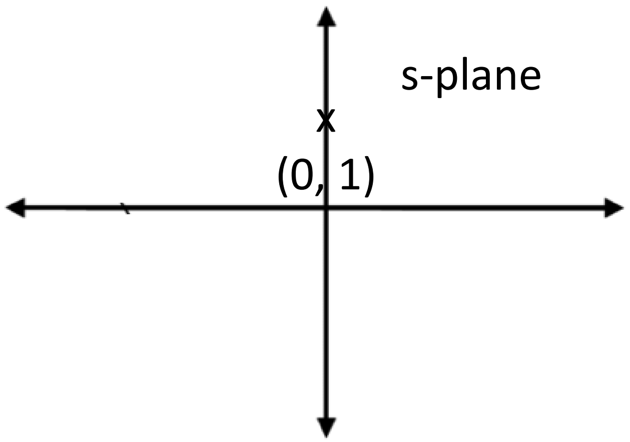
$$z = e^{\alpha T}(\cos\omega T + j\, \sin\omega T) = \boxed{e^{\alpha T}\angle \omega T}$$

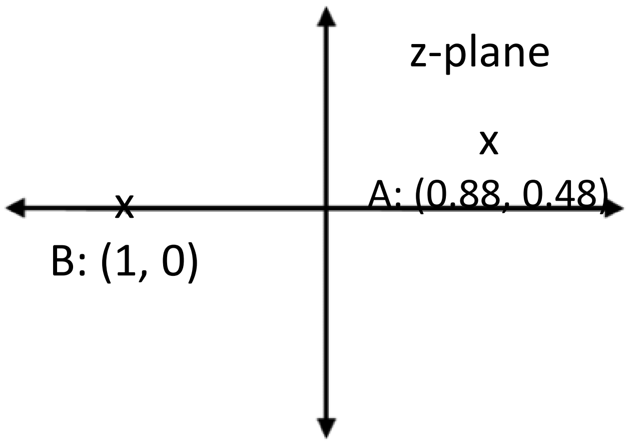
- A point or region on s-plane can be mapped into a corresponding point or region on z-plane.

Question: If T=0.5, which point on z-plane is mapped from the point (2,0) on s-plane? How about (0,1)?



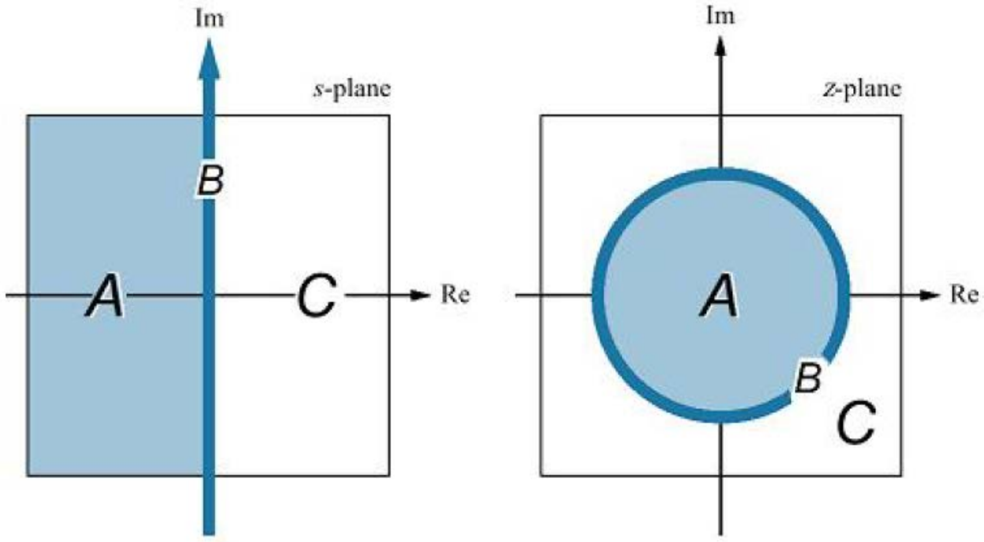




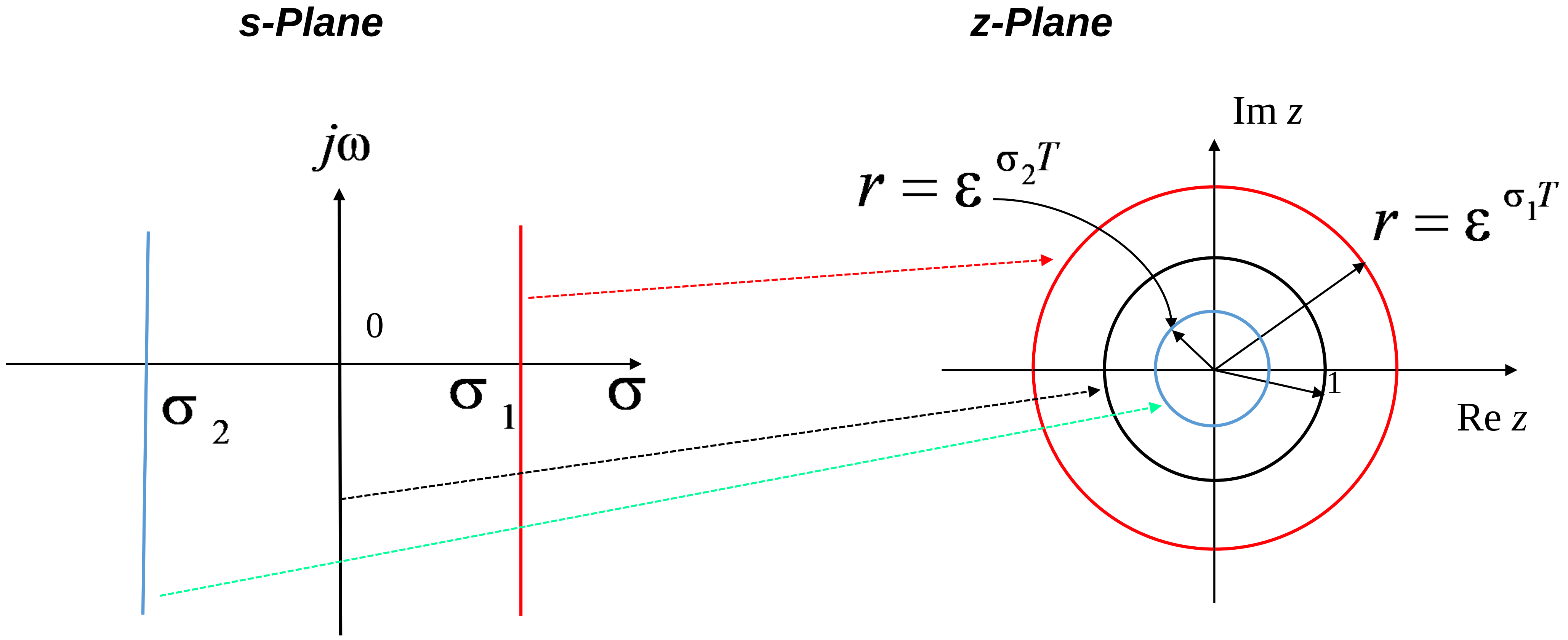


We assume $s = \alpha + j\omega$

- Points on the $j\omega$-axis, α have zero values, so $e^{\alpha T}$ = 1. Thus points on the $j\omega$-axis of s-plane map into points on the unit circle on z-plane.
- If α is positive, $e^{\alpha T}$ > 1. Thus points on the positive real axis of s-plane map into points outside the unit circle on z-plane.
- If α is negative, $e^{\alpha T}$< 1. Thus points on the negative real axisof s-plane map into points inside the unit circle on z-plane.

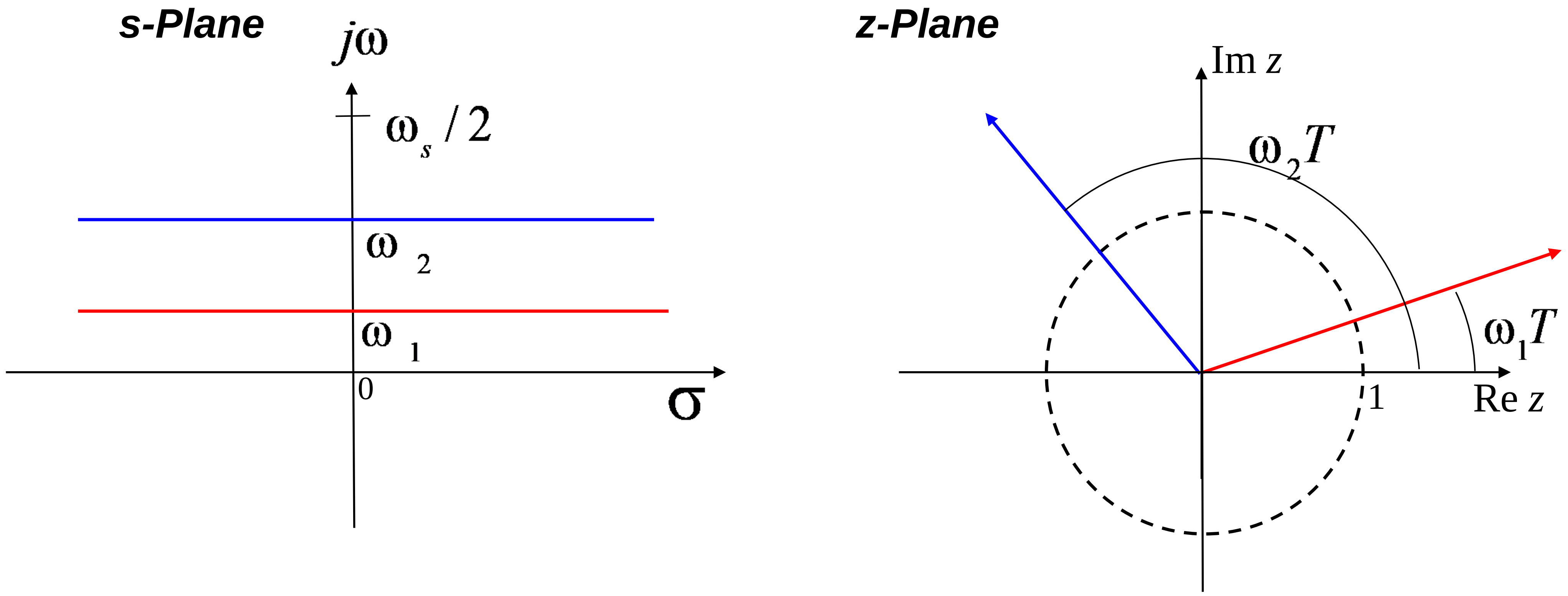


Mapping from s-plane to z-plane



Vertical lines on the s-plane map to circles with constant radius on the *z*-plane.

Mapping from s-plane to z-plane



Horizontal lines on the *s*-plane map to lines of constant angle on the z-plane.

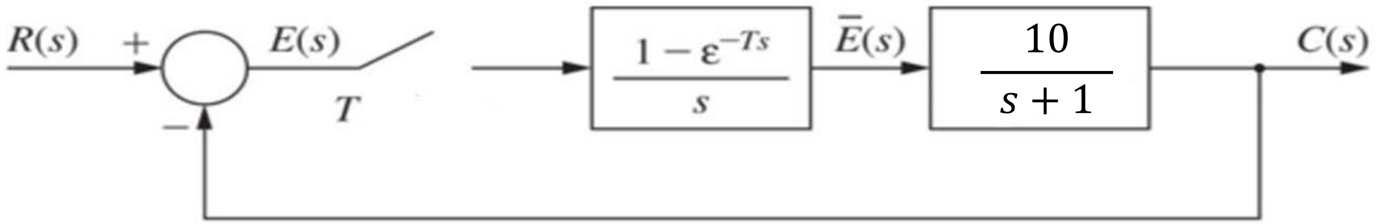
Stability of Digital Control System

- Stable, if all poles of closed loop transfer function are inside the unit circle on z-plane.

- Unstable, if any of the poles is outside the unit circle.

- Marginally stable, if one or more poles are on the unit circle and all other poles are inside the unit circle.

Example



What T value makes the system stable?

- $G(s) = \frac{1-e^{-sT}}{s}\frac{10}{s+1} = 10 \times \left(1 - e^{-sT}\right)[\frac{1}{s} - \frac{1}{s+1}]$ $\quad G(z) = \frac{z-1}{z} Z[\frac{10}{s(s+1)}]$
- Taking z-transfer function:

$$G(z) = 10 \times \left(1 - z^{-1}\right)\left[\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right] = 10\,\frac{1-e^{-T}}{z-e^{-T}}$$

- Closed-loop TF in z-plane is

$$T(z) = \frac{G(z)}{1+G(z)} = \frac{10(1-e^{-T})}{z-(11e^{-T}-10)}$$

- The pole is

$$z = (11e^{-T} - 10)$$

Value of poles when T changes

$$z = (11e^{-T} - 10)$$

- $T = 0$ ➔ $z = 1$
- $T = 0.1$ ➔ $z = -0.05$
- $T = 0.2$ ➔ $z = -1$
- $T = 0.3$ ➔ $z = -1.85$
- …

The system is stable for $T \leq 0.2$

Root Locus on Z-Plane

Determines the location of roots of the characteristic equation of a closed loop control system as the overall system gain (often denoted by K) varies.

Steps to plot closed-loop poles

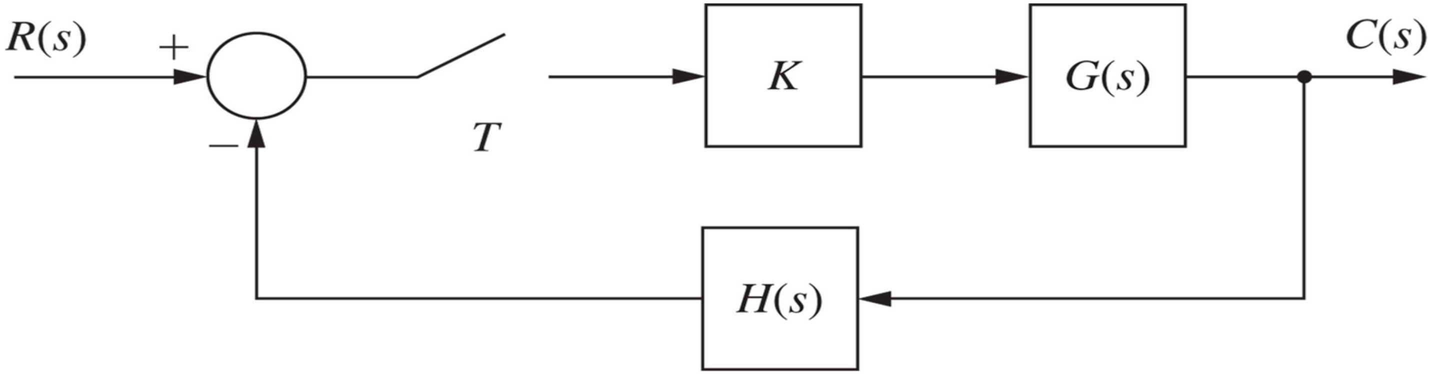
1. Derive the open loop function $K\overline{GH}$.

2. Factor numerator and denominator to get open loop zeros and poles.

3.Plot roots of $1 + K\overline{GH} = 0$ in *z*-Plane as K varies.

Note that while construction rules for the *z*-plane are identical with those for the *s*-Plane.

Consider Sampled-data System



- The closed-loop transfer function is $\frac{K\ G(z)}{1+KZ[G(s)H(s)]} \triangleq \frac{K\ \mathrm{G}(z)}{1+K\overline{GH}}$
- System characteristic equation is $1 + K\overline{GH} = 0$, and $K\overline{GH}$ is open loop TF

Rules To Plot Root Locus

$$1 + K\overline{GH}(z) = 0 \qquad \overline{GH}(z) = \frac{N(z)}{D(z)}$$

1. Loci originate on the poles of $K\overline{GH}$ and terminate on its zeros.

2. The loci are <u>symmetrical</u> with respect to the real axis.

3. The number of asymptotes is equal to the number of poles of $K\overline{GH}$, n_p, minus the number of its zeros, n_z. The angles of the asymptotes are found by $\theta_a = \frac{(2k+1)\pi}{n_p - n_z}$, $k = 0, 1, 2, \ldots \left(n_p - n_z - 1\right)$, where n_p is # of finite poles and n_z is # of finite zeros.

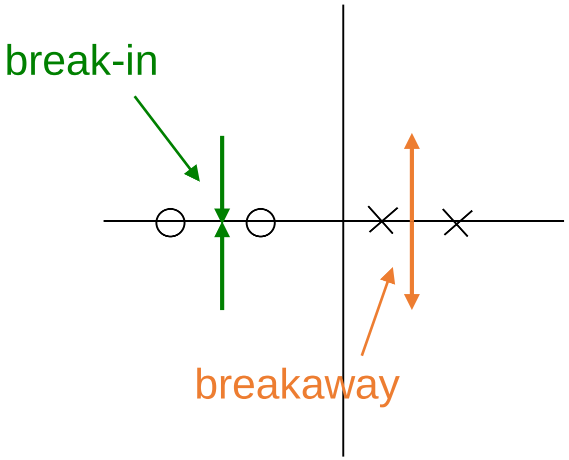
Rules to Plot Root Locus

4. The origin of the asymptotes on the real axis is given by:

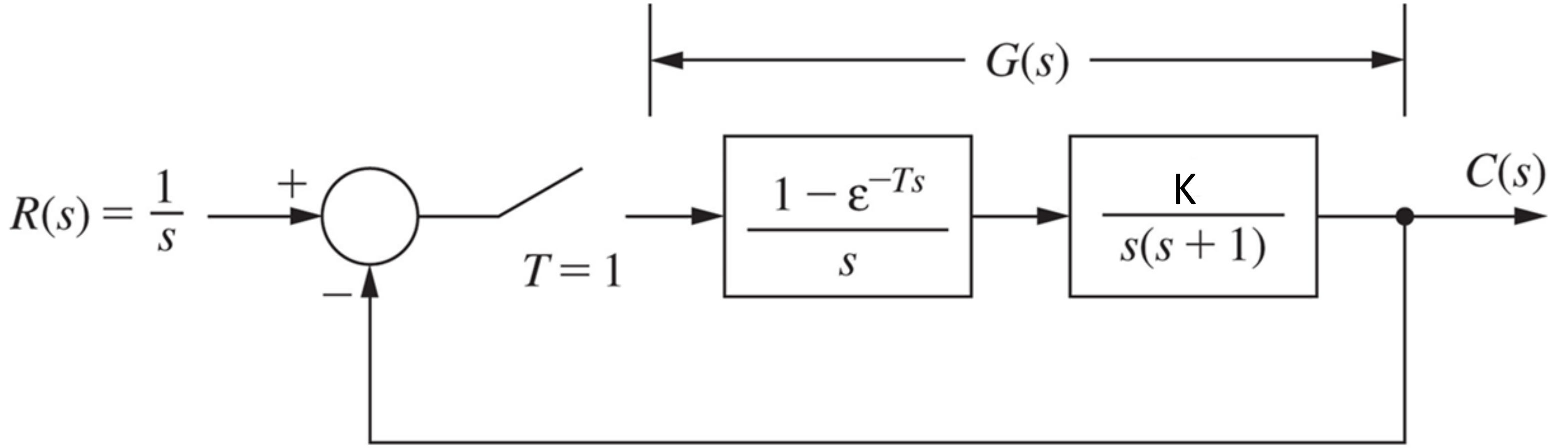
$$\sigma = \frac{\sum \text{poles of } \overline{GH}(z) - \sum \text{zeros of } \overline{GH}(z)}{n_p - n_z}$$

5. The breakaway point for the locus between two poles (or the break-in point for the locus between two zeros) is found by

$$\frac{d[\overline{GH}(z)]}{dz} = 0$$



Root Locus Example



- Consider the system, we have

$$G(z) = \left(1 - z^{-1}\right) Z\left[\frac{K}{s^2(s+1)}\right]_{(T=1)} = K\frac{z-1}{z}\left[\frac{z^2e^{-1} + z - 2ze^{-1}}{(z-1)^2(z-e^{-1})}\right] = \frac{K(ze^{-1} + 1 - 2e^{-1})}{(z-1)(z-e^{-1})}$$

- Hence, the open loop transfer function $G(z) = \frac{0.368K(z+0.718)}{(z-1)(z-0.368)}$

Open-loop TF: $G(z) = \frac{0.368K(z+0.718)}{(z-1)(z-0.368)}$

Plot the Locus of Roots

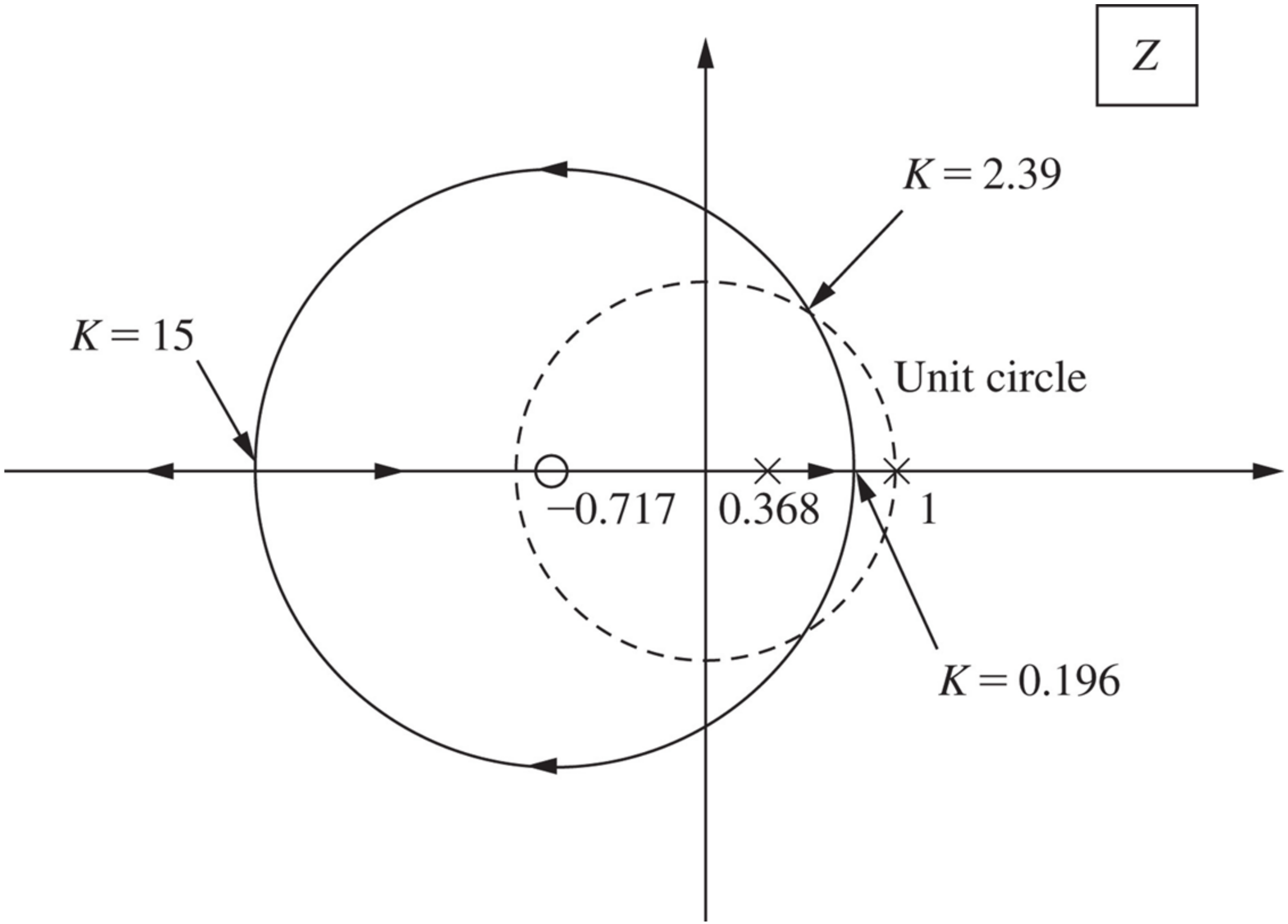
(1) Start at z=1 and z=0.368, and terminate at z=-0.718 and z=infinity

(2) Break-away or break-in points: $\frac{d}{dz}G(z) = 0$ occurs at $z = 0.64$ and $z = -2.08$

$$\frac{d}{dz}G(z) = \frac{0.368K}{(z-1)(z-0.368)} + \frac{(-1)0.368K(z+0.718)}{[(z-1)(z-0.368)]^2}(2z-1.368) = 0$$

➔ $\mathrm{z}^2 + 1.436z - 1.35 = 0$ ➔ $z = 0.64$ or $z = -2.08$

What is the corresponding K value for $z = 0.64$ or $z = -2.08$?


Z
K = 2.39
K = 15
Unit circle
−0.717
0.368
1
K = 0.196

What gain value of K yields break-in and break-away points?

Open-loop Transfer Function

$$G(z) = \frac{0.368K(z + 0.718)}{(z - 1)(z - 0.368)}$$

- $z = 0.64$ or $z = -2.08$ are the poles of close loop system

  The characteristic equation is $1 + G(z) = 0$. Hence,

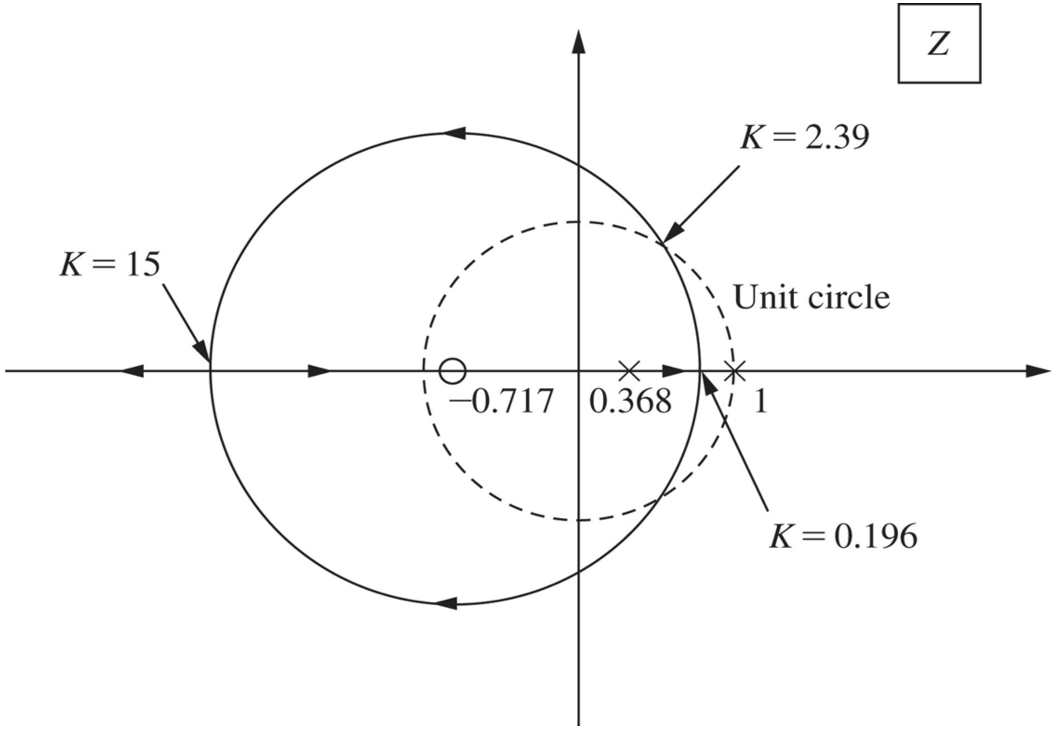
$$0.368\mathrm{K}(\mathrm{z} + 0.718) + (z - 1)(z - 0.368) = 0$$

- Substitute $z = 0.64$ or $z = -2.08$, and we get

$K = 0.20$ at $z = 0.64$

and $K = 15.0$ at $z = -2.08$

The K value achieving marginally stable makes the root locus $1 + G(z) = 0$ cross the unit circle.

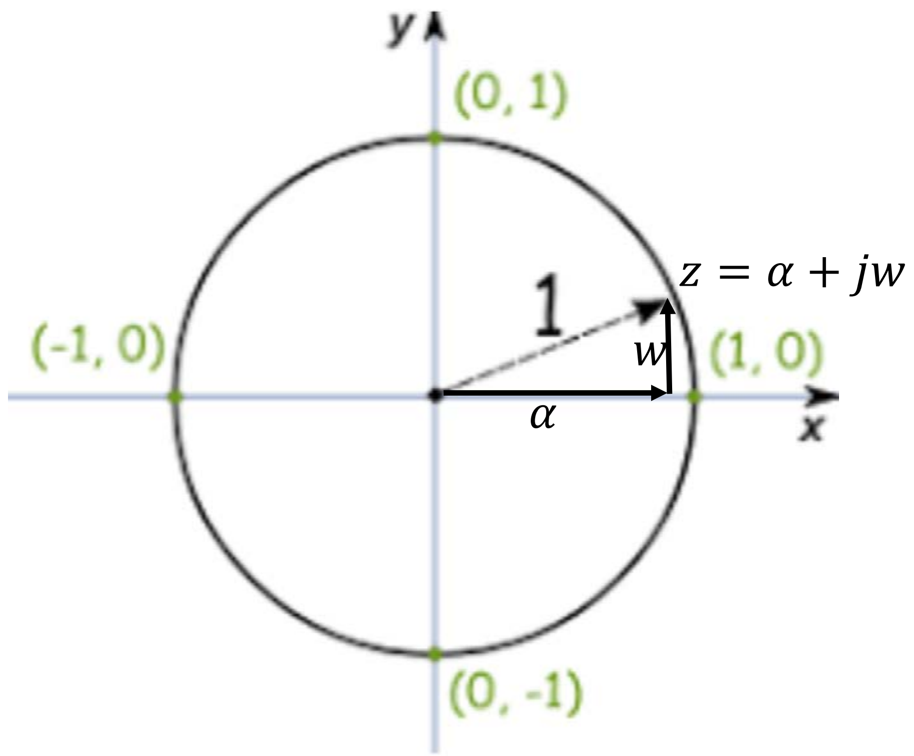


- K value that makes the roots appear on the unit circle implies the gain value for stability.

- So K satisfies:

$$0.368\mathrm{K}(\mathrm{z} + 0.718) + (z - 1)(z - 0.368) = 0$$
$$|z| = 1$$

$$K = 2.39$$



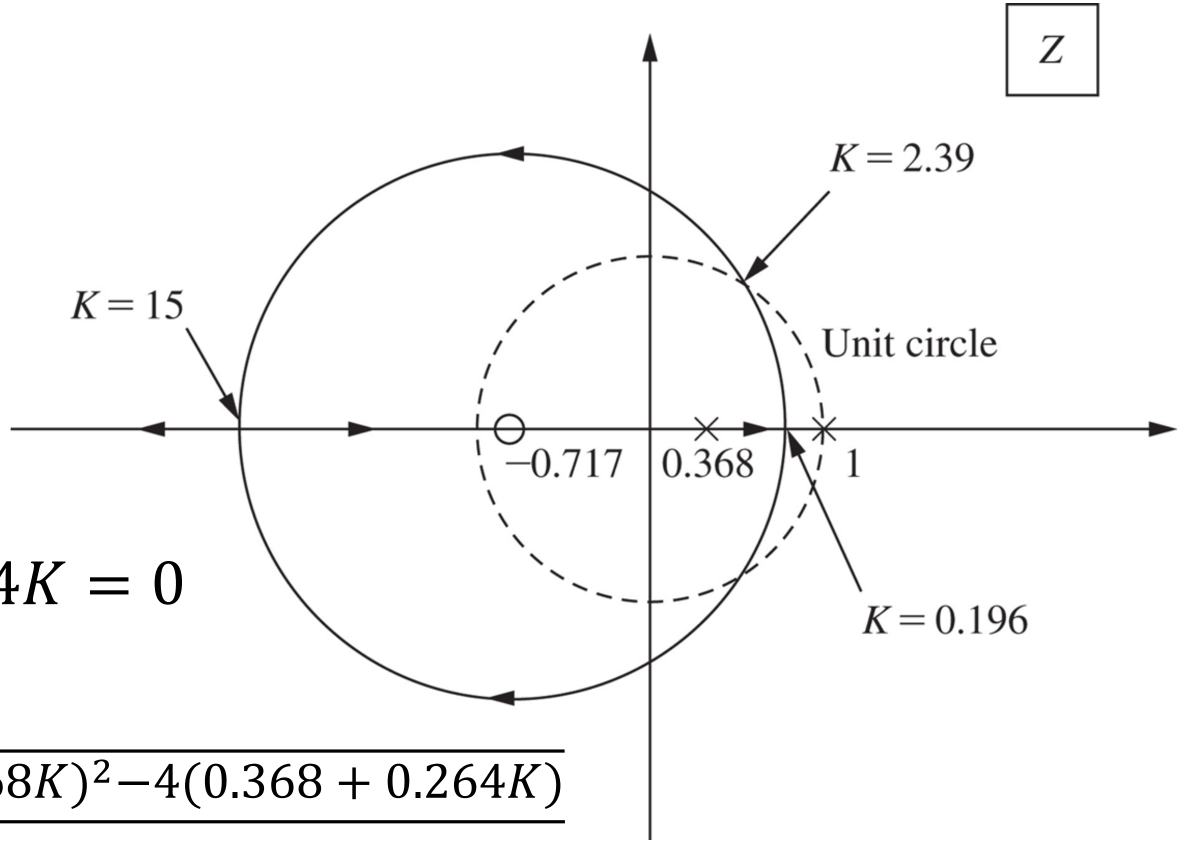
If $z = \alpha + jw$ is on the unit circle, we must have $\alpha^2 + w^2 = 1$

(Not $\alpha^2 + (jw)^2 = 1$)

Continue…

Open-loop TF: $G(z) = \frac{0.368K(z+0.718)}{(z-1)(z-0.368)}$

How to determine the value of K, which yields marginally stable?



Characteristic equation:

$$z^2 + (0.368K - 1.368)z + 0.368 + 0.264K = 0$$

$$z = \frac{(1.368 - 0.368K) \pm \sqrt{(1.368 - 0.368K)^2 - 4(0.368 + 0.264K)}}{2}$$

By forcing the magnitude of z to be one, so $\frac{(1.368-0.368K)^2 + \left[4(0.368+0.264K) - (1.368-0.368K)^2\right]}{4} = 1 \Rightarrow K = 2.39.$

End!