Real Time Systems and Control Applications

Contents

Convert from CCS to DCS

Digital Controller in Continuous Control System

- In reality, the plant to be controlled is likely to be a continuous control system (CCS).
- We have developed pretty mature theories and techniques to analyze and design a continuous controller.
- What if we want to use digital controller instead of continuous controller?

If we know G(s), how to implement a digital controller y(k) = ?
y(k) is a function of current and previous inputs u(k), u(k-1), u(k-2),… and previous output y(k-1), y(k-2)…

G(s) → G(z) → y(k)

- G(s) → G(z)

Example: $G(s) = \frac{s+3}{s^2+6s+8}$ ➔ $G(z) = \frac{1}{2}(\frac{z}{z-e^{-2T}} + \frac{z}{z-e^{-4T}})$

- G(z) → y(k)

$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{2}\frac{z(z-e^{-2T}+z-e^{-4T})}{(z-e^{-2T})(z-e^{-4T})} \xRightarrow{T=1} \frac{z(z-0.0765)}{(z-0.135)(z-0.018)}$$

$$\frac{Y(z)}{U(z)} = \frac{z^2-0.0765z}{z^2-0.153z+0.00243}$$

Continue…

$$\frac{Y(z)}{U(z)} = \frac{z^2 - 0.0765z}{z^2 - 0.153z + 0.00243}$$

$$y(k) = 0.153y(k-1) - 0.00243y(k-2) + u(k) - 0.0765u(k-1)$$

Recall: A delay in the time domain corresponds to the z-transform of the signal without delay, multiplied by a power of z:

$$u(k-1) \leftrightarrow z^{-1}U(z)$$

Generally,

$$u(k-n) \leftrightarrow z^{-n}U(z)$$

What if we don't know the roots of denominator of $G(s)$?

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

CCS (Continuous Control System) to DCS (Digital Control System)

- Assume the transfer function of a continuous controller is given by

$$D(s) = \frac{U(s)}{E(s)} = K_0 \frac{s+a}{s+b}$$

where U(s) is the transfer function of the output u(t) and E(s) is the transfer function of the input (error signal) e(t) to the controller.

The objective is to find the difference equations to be programmed into a computer, then determine the software implementation of the controller which approximate the original continuous controller.

Obtain Difference Equation

$$D(s) = \frac{U(s)}{E(s)} = K_0 \frac{s+a}{s+b}$$

- Cross multiplication gives:

$$U(s)(s+b) = K_0 E(s)(s+a)$$

sU(s) → u'(t) if U(s) → u(t)

Hence, $u'(t) + bu(t) = K_0 e'(t) + aK_0 e(t)$

- Recall Euler's Method,

$$u' = \lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \lim_{T \to 0} \frac{u(kT+T) - u(kT)}{T}$$

- If sampling period T is small enough, the above expression can be approximated to $u' = \frac{u(kT+T) - u(kT)}{T}$.

- By letting $u(kT+T)=u(k+1)$ the value of u at the time interval t_{k+1}, $u' \cong \frac{u(k+1)-u(k)}{T}$.

- Hence, $u'(t)+bu(t)=K_0e'(t)+aK_0e(t)$ can be expressed as

$$\frac{u(k+1)-u(k)}{T}+bu(k)=K_0\frac{e(k+1)-e(k)}{T}+aK_0e(k)$$

- Rearranging to get:

$$u(k+1)=(1-bT)u(k)+K_0(aT-1)e(k)+K_0e(k+1)$$

$$u(k)=(1-bT)u(k-1)+K_0(aT-1)e(k-1)+K_0e(k)$$

It shows that a new value of the output at t_{k+1} can be computed from the past value of the control $u(k)$ and the now and past values of the error signal $e(k)$, $e(k+1)$.

Z-Transform Of Difference Equation

- Given $u(k+1) = (1-bT)u(k) + K_0(aT-1)e(k) + K_0 e(k+1)$, what is the corresponding z-transform?

$$\frac{U(z)}{E(z)} = \frac{K_0(aT-1)z^{-1} + K_0}{1+(bT-1)z^{-1}} = \frac{K_0 z + K_0(aT-1)}{z+(bT-1)}$$

$$u(k+1) - (1-bT)u(k) = K_0(aT-1)e(k) + K_0 e(k+1)$$
$$zU(z) - (1-bT)U(z) = K_0(aT-1)E(z) + zK_0E(z)$$
$$U(z)[\mathrm{z}-(1-bT)] = E(z)[K_0(aT-1) + zK_0]$$
$$U(z)/E(z) \ = [K_0(aT-1) + zK_0]/[\mathrm{z}+(bT-1)]$$

Another Example

- $D(s) = \frac{U(s)}{E(s)} = \frac{a}{s+a}$ u(kT)=u(k)

  $U(s)(s + a) = aE(s)$ ➔ Laplace Transform ➔ $\frac{u(k+1)-u(k)}{T} + a\,u(k) = a\,e(k)$

- The difference equation is:

$$u(k + 1) = (1 - aT)u(k) + aTe(k)$$

- The corresponding z-transform is:

$$\frac{U(z)}{E(z)} = \frac{aTz^{-1}}{1 + (aT - 1)z^{-1}} = \frac{aT}{z + (aT - 1)}$$

Now Consider Z-Transform

$$D(z) = \frac{U(z)}{E(z)} = \frac{aT}{z + (aT - 1)}$$

- System is given:

$$D(s) = \frac{U(s)}{E(s)} = \frac{a}{s + a}$$

- We know the Inverse Laplace Transform:

$$d(t) = ae^{-\mathrm{at}}$$

Different?

- The z-Transform gives:

$$D(z) = \frac{U(z)}{E(z)} = \frac{az}{z - e^{-aT}}$$

Discrete Equivalents via Numerical Integration

- Consider, for example:

$D(s) = \frac{U(s)}{E(s)} = \frac{a}{s+a}$ ➔ U(s)s+aU(s)=aE(s)➔u'(t)=-au(t)+ae(t)

- Differential equation: $u' = -au + ae$

$$u' \cong \frac{u(k+1) - u(k)}{T}$$

- Hence,

$$u(t) = \int_0^t [-au(\tau) + ae(\tau)]d\tau$$

For Discrete System

$$u(kT) = \int_{0}^{kT-T} [-au(\tau) + ae(\tau)]d\tau + \int_{kT-T}^{kT} [-au(\tau) + ae(\tau)]d\tau$$

$$u(kT) = u(kT - T) + \int_{kT-T}^{kT} [-au(\tau) + ae(\tau)]\, d\tau$$

- The second term can be approximated as the area of $-au + ae$ for $kT - T \leq \tau \leq kT$. There are many rules to approximate the incremental area term.

Approximation Methods

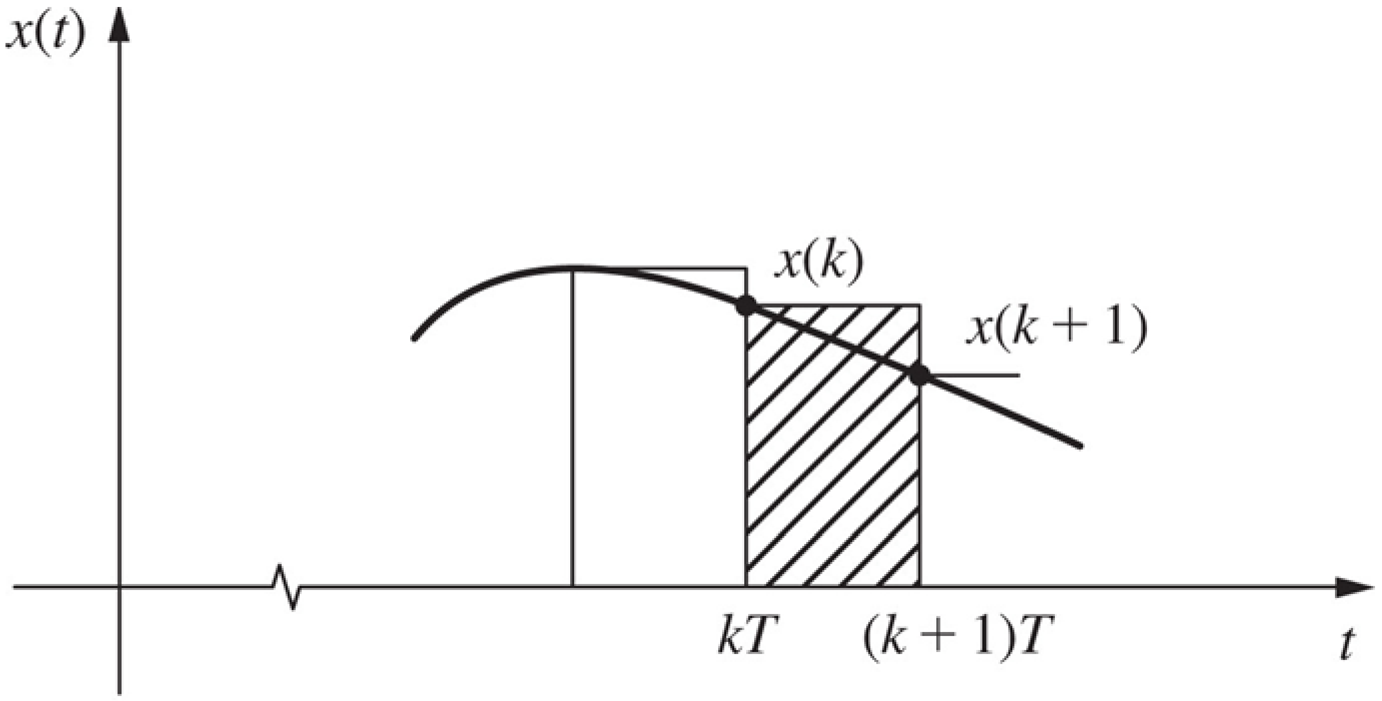
- Forward Rectangular Rule

- Backward Rectangular Rule

- Trapezoid Rectangular Rule or Tustin's Method or Bilinear Transformation

Each of these techniques can be used to find the discrete transfer function and difference equation for a controller, if the continuous transfer function is known.

Forward Rule for Numerical Integration



Forward Rectangular Rule

- In this case, area under the curve is approximated by area of the rectangular looking forward from $kT - T$. The height of rectangle is the amplitude of the curve $(-au + ae)$ at $kT - T$ and the width is T. This results in

$$u(kT) = u(kT - T) - aTu(kT - T) + aTe(kT - T)$$

$$\rightarrow\ u(kT) = (1 - aT)u(kT - T) + aTe(kT - T)$$

$$\rightarrow\ U(z) = (1 - aT)z^{-1}U(z) + aTz^{-1}E(z)$$

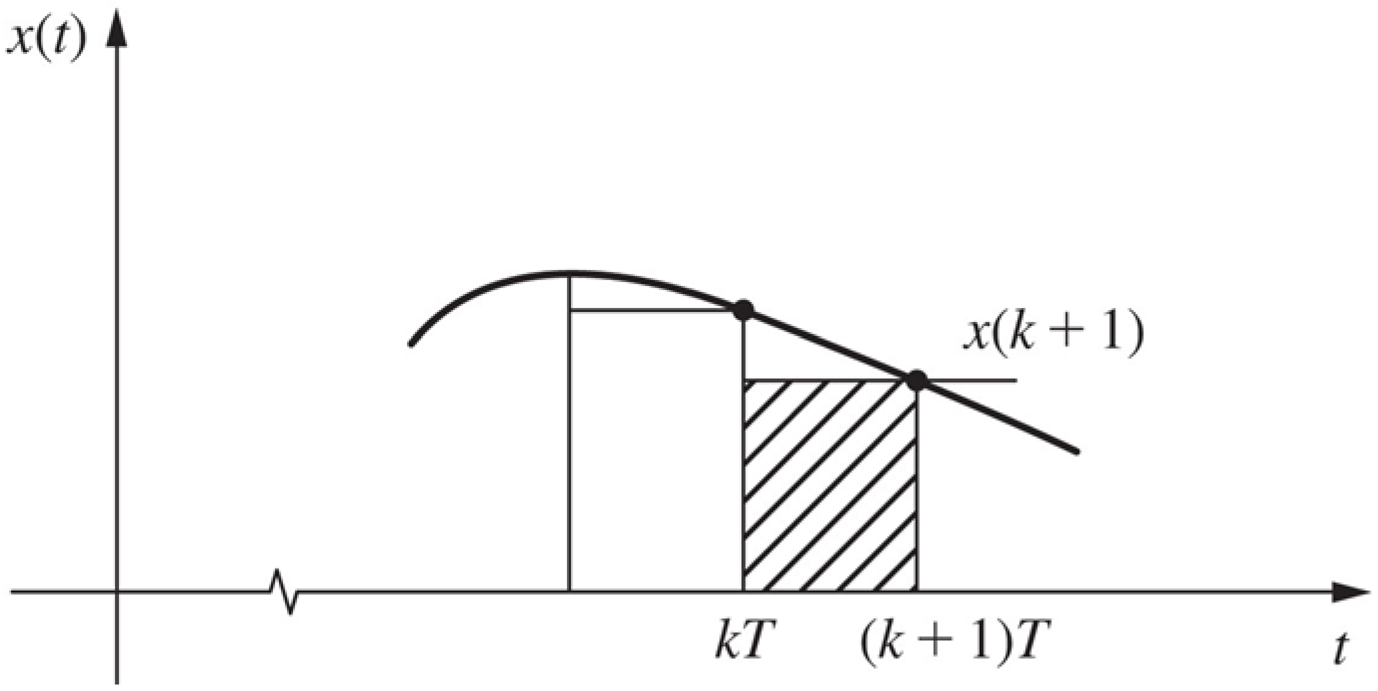
Collecting like terms to have:

$$U(z)\left[1 - (1 - aT)z^{-1}\right] = aTz^{-1}E(z)$$

Hence,

$$\frac{U(z)}{E(z)} = \frac{aTz^{-1}}{1 - (1 - aT)z^{-1}} = \frac{a}{\frac{z-1}{T} + a}$$

Backward Rule for Numerical Integration



Backward Rectangular Rule

- This approximation takes the amplitude of the rectangle as the value of $(-au + ae)$ at kT (i. e. looking backward from kT towards $kT - T$. The equation for this approximation becomes:

$$u(kT) = u(kT - T) - aTu(kT) + aTe(kT)$$

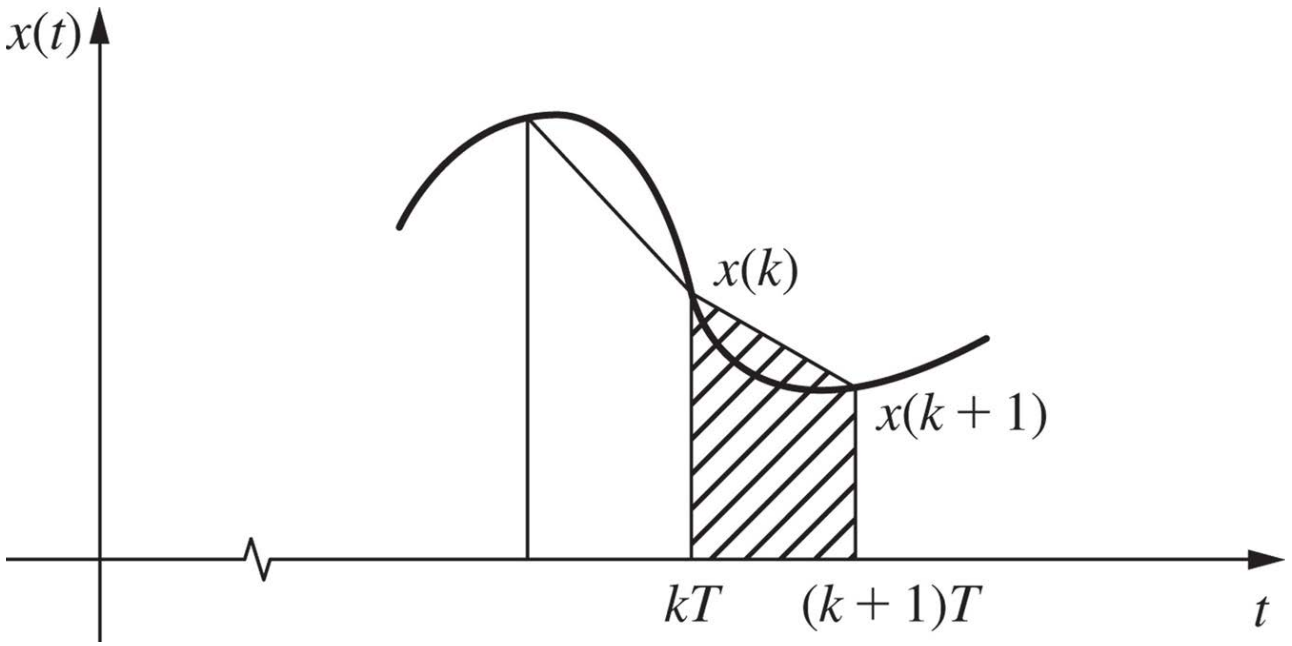
$$\rightarrow \ (1 + aT)u(kT) = u(kT - T) + aTe(kT)$$

$$\rightarrow \ [(1 + aT) - z^{-1}]U(z) = aTE(z)$$

Hence,

$$\frac{U(z)}{E(z)} = \frac{aT}{(1 + aT) - z^{-1}} = \frac{aTz}{z + aTz - 1} = \frac{a}{\frac{z - 1}{Tz} + a}$$

Trapezoidal Rule for Numerical Integration



Trapezoid Rectangular Rule or Tustin's Method or Bilinear Transformation

- This rule considers the incremental area to be that of the trapezoid formed by average of rectangles used by previous two rules giving:

$$u(kT) = u(kT - T) + \frac{T}{2}[-aTu(kT - T) + aTe(kT - T) - aTu(kT) + aTe(kT)]$$

- Hence,

$$u(kT) = \frac{1 - \left(\frac{aT}{2}\right)}{1 + \left(\frac{aT}{2}\right)} u(kT - T) + \frac{\frac{aT}{2}}{1 + \left(\frac{aT}{2}\right)} [e(kT - T) + e(kT)]$$

- The corresponding transfer function is:

$$\frac{U(z)}{E(z)} = \frac{aT(z + 1)}{(2 + aT)z + aT - 2} = \frac{a}{\left(\frac{2}{T}\right)[(z - 1)/(z + 1)] + a}$$