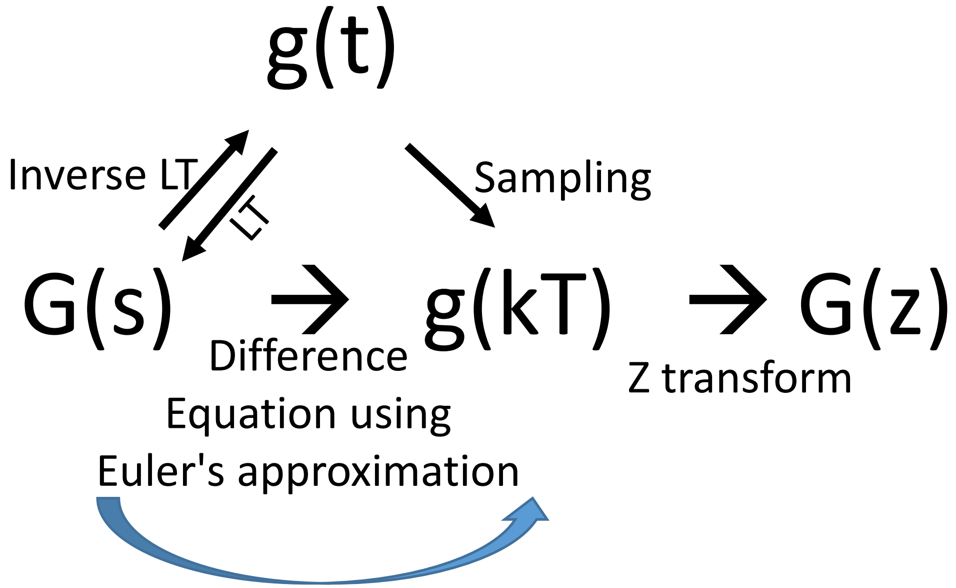
Several Announcements

- Make-up Lab in this week.
- Final Exam on Dec. 18th   9am—11am
- I will give a review for the final exam on Thursday (Nov. 28th).
- The last quiz section will be on Nov. 29th.
- I will host office hours on Dec. 5th, 12th from 10:00 to 12:00 at ITB214, or by appointment.

Real Time Systems and Control Applications

Contents

$G(s) \rightarrow G(z)$



Numerical Integration:
Forward rule
Backward rule
Trapezoidal rule

Three Rules to Approximate D(z) from D(s)

A direct comparison shows that given $D(s)$, the corresponding $D(z)$ can be obtained by substitution of an approximation for the frequency variable s, as shown below:

D(s)	Rule	Z-transfer function D(z)
$\frac{a}{s+a}$	Forward	$\frac{a}{(z-1)/T+a}$
$\frac{a}{s+a}$	Backward	$\frac{a}{(z-1)/(Tz)+a}$
$\frac{a}{s+a}$	Trapzoid	$\frac{a}{(2/T)[(z-1)/(z+1)]+a}$

Three Rules to Approximate D(z) from D(s)

Method	Approximation
Forward Rule	$s \leftarrow \frac{z-1}{T}$
Backward Rule	$s \leftarrow \frac{z-1}{Tz}$
Trapezoidal Rule	$s \leftarrow \frac{2}{T}\frac{z-1}{z+1}$

G(s) → G(z) → Software implementation g(k)

Example

$$D(s) = \frac{U(s)}{E(s)} = 70\frac{s+2}{s+10}$$

a) Derive the corresponding differential equation.

b) Use Euler's approximation to determine the difference equation for a digital implementation with a sampling rate of 20 Hz.

c) Substitute the expression for the frequency variable s corresponding to forward rectangular rule, in the transfer function and derive D(z).

d) Use D(z) to derive difference equation for the compensator and compare the result with that of part (b)

Answer:

- (a) $u' + 10u = 70(e' + 2e)$

- (b) Euler's approximation to determine the difference equation for a digital implementation with a sampling rate of 20Hz.

$$\frac{u(k+1) - u(k)}{T} + 10u(k) = 70[\frac{e(k+1) - e(k)}{T} + 2e(k)]$$

$$u(k+1) = (1 - 10T)u(k) + 70(2T - 1)e(k) + 70e(k+1)$$

For 20 Hz sampling rate, $T = \frac{1}{20} = 0.05$

$$u(k+1) = 0.5u(k) + 70e(k+1) - 63e(k)$$

- (c) According to forward rectangular rule,

$$s \leftarrow \frac{z-1}{T}$$

$$D(z) = 70\,\frac{\frac{z-1}{T} + 2}{\frac{z-1}{T} + 10}$$

- For $T = 0.05$,

$$D(z) = \frac{70z - 63}{z - 0.5}$$

- (d)

$$D(z) = \frac{U(z)}{E(z)} = \frac{70z - 63}{z - 0.5} = \frac{70 - 63z^{-1}}{1 - 0.5z^{-1}}$$

$$(1 - 0.5z^{-1})U(z) = \left(70 - 63z^{-1}\right)E(z)$$

$$U(z) = 0.5z^{-1}U(z) + 70E(z) - 63z^{-1}E(z)$$

Hence,

$$u(k) = 0.5u(k-1) + 70e(k) - 63e(k-1)$$

The two results in (b) and (d) are same, as both say that the current value of output depends upon its previous value and the current and the previous values of the error signal.

Example

- For the transfer function $Gc(s) = 2/(s + 4)$, obtain the discrete transfer function by means of the forward, backward and trapezoidal rectangular rules, using a sampling period $T = 0.1s$

For transfer function $G_c = \frac{2}{s+4}$ find discrete transfer function by forward rectangular rule

Substituting $s = \frac{z-1}{T}$ for forward rectangular rule gives:

$$G(z) = \frac{2}{\frac{z-1}{0.1} + 4}$$

$$= \frac{0.2}{z - 1 + 0.4} = \frac{0.2}{z - 0.6}$$

For transfer function $G_c = \frac{2}{s+4}$ find discrete transfer function by backward rectangular rule

Substituting $s = \frac{z-1}{zT}$ for backward rectangular rule results in:

$$G(z) = \frac{2}{\frac{z-1}{zT} + 4} = \frac{2Tz}{z(4T+1) - 1}$$

For transfer function $G_c = \frac{2}{s+4}$ find discrete transfer function by trapzoidal rectangular rule

Substituting $s = \frac{2}{T}\frac{z-1}{z+1}$ for trapzoidal rectangular rule:

$$G(z) = \frac{2}{\frac{2}{T}\frac{z-1}{z+1} + 4} = \frac{T(z+1)}{(2T+1)z + (2T-1)}$$

Conversion from s to z plane	
Forward Rule	$s \leftarrow \frac{z-1}{T}$
Backward Rule	$s \leftarrow \frac{z-1}{Tz}$
Trapezoidal Rule	$s \leftarrow \frac{2}{T}\frac{z-1}{z+1}$

Map from z-plane to s-plane

Method	Approximation
Forward Rule	$z \leftarrow sT + 1$
Backward Rule	$z \leftarrow \frac{1}{1 - Ts}$
Trapezoidal Rule	$z \leftarrow \frac{1 + Ts/2}{1 - Ts/2}$

Each relations can be viewed as a map from s-plane to z-plane.
Since we are more familiar with s-plane characteristics, it would be useful to find the inverse of each transformation as a map from s-plane to z-plane.

View The Approximation From Another Angle

- In z-transform we have

$$z = e^{sT}$$

- Consider power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- Then, we consider the linear approximation

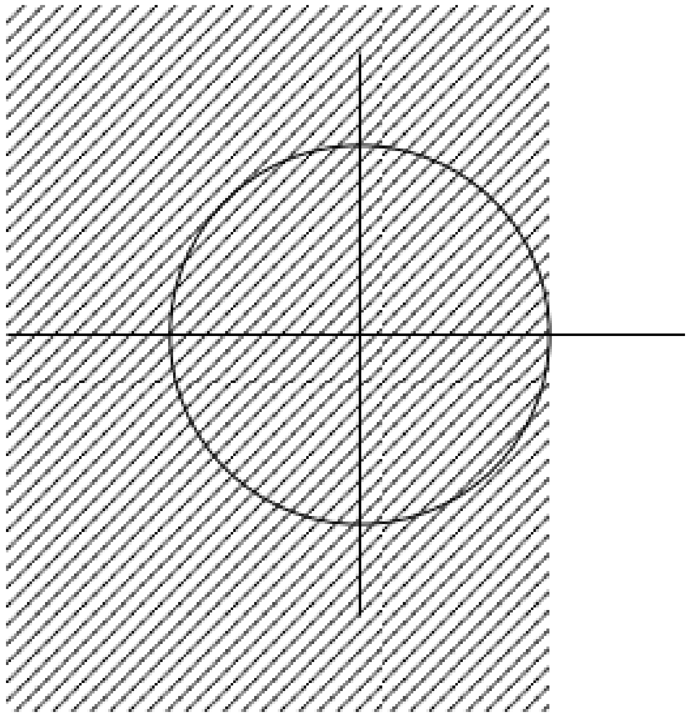
$$z = e^{sT} = 1 + sT$$

$$z = \frac{1}{e^{-sT}} = \frac{1}{1 - sT}$$

$$z = \frac{e^{\frac{1}{2}sT}}{e^{-\frac{1}{2}sT}} = \frac{1 + {sT}/{2}}{1 - {sT}/{2}}$$

Stability Regions by Forward Rectangular Mapping Rules

- Forward Rectangular Rule: $z \leftarrow sT + 1$



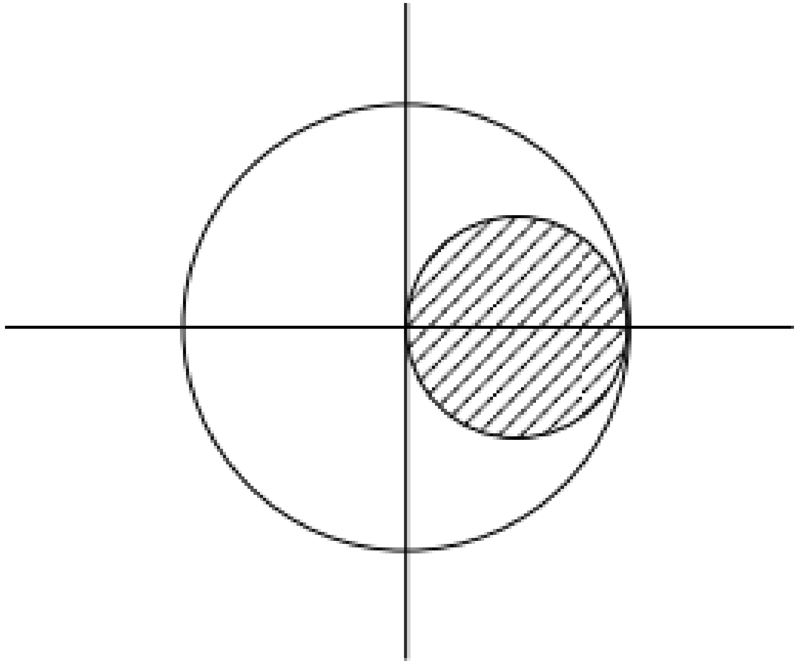
If the discrete system is stable, then the corresponding continuous system is stable under forward rectangular rule.

It is possible that continuous system is stable, but the corresponding discrete system is unstable.

Discrete stable → Continuous stable

Stability Regions by Backward Rectangular Mapping Rules

- Backward Rectangular Rule: $z \leftarrow \frac{1}{1-Ts}$



Consider $z = \frac{1}{1-Ts} = \frac{1}{2} + \left(\frac{1}{1-Ts} - \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2}\frac{1+Ts}{1-Ts}$

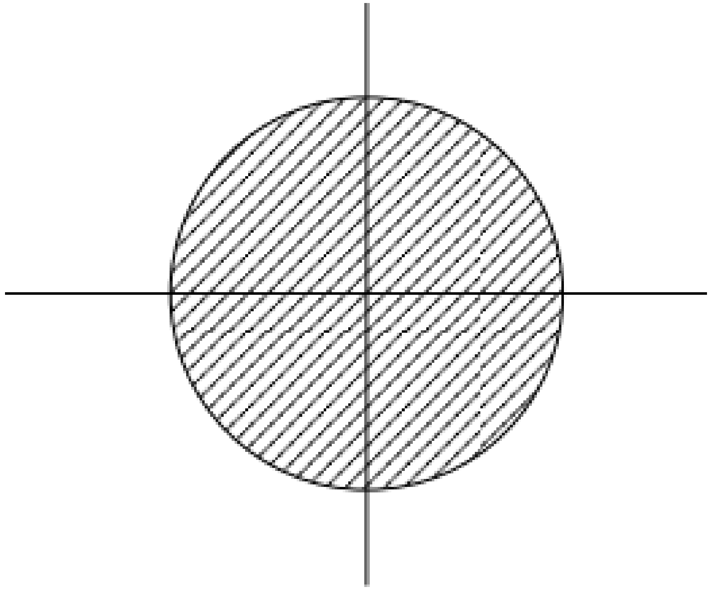
Hence, we write $|z - \frac{1}{2}| = \frac{1}{2}$

It is possible that discrete system is stable, but the corresponding continuous is unstable.

Discrete stable ← Continuous stable

Stability Regions by Trapezoidal Rectangular Mapping Rules

- Trapezoidal Rectangular Rule: $z \leftarrow \frac{1+Ts/2}{1-Ts/2}$ (bilinear transformation).



Under trapezoid rule the stable region of s-plane is mapped exactly to the stable region of z-plane.

Continuous system stable ↔ Discrete system stable

Bilinear transformation maps imaginary axis in s-plane to the unit circle in the z-plane

$z \leftarrow \frac{1+Ts/2}{1-Ts/2}$

When s is on the imaginary axis

$|z| = 1$

Proof: Having $s = j\Omega$, we have

$$\frac{1 + Ts/2}{1 - Ts/2} = \frac{1 + T(j\Omega)/2}{1 - T(j\Omega)/2}$$

Hence,

$$z \leftarrow \frac{1+j\omega}{1-j\omega} \quad \text{where } \omega = \frac{T}{2}\Omega$$

Bilinear transformation maps the inside of the unit circle in the z-plane into the left half of the s-plane

- By forcing $\omega = Ts/2$

$$|z| = \left|\frac{w+1}{w-1}\right| = \left|\frac{\alpha + j\beta + 1}{\alpha + j\beta - 1}\right| < 1$$

$$\Rightarrow \frac{(\alpha+1)^2 + \beta^2}{(\alpha-1)^2 + \beta^2} < 1 \Rightarrow (\alpha+1)^2 + \beta^2 < (\alpha-1)^2 + \beta^2 \Rightarrow \alpha < 0$$

Revisit System $D(s) = \frac{U(s)}{E(s)} = \frac{a}{s+a}$

- $u(t) = 1 - e^{-at}$

- By Forward Rule:

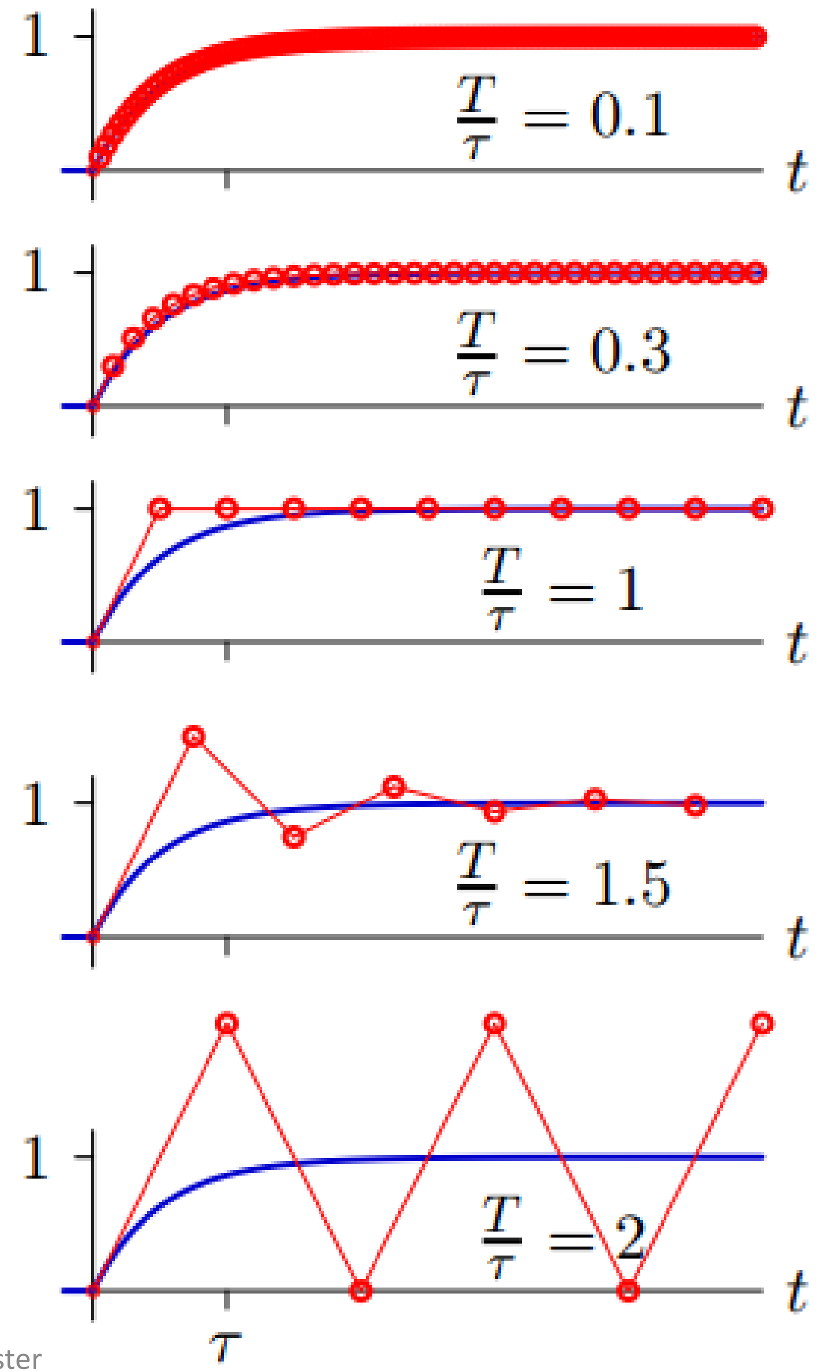
$$u(kT) = (1 - aT)u(kT - T) + aTe(kT - T)$$

- By Backward Rule:

$$(1 + aT)u(kT) = u(kT - T) + aTe(kT)$$

$$u(kT) = (1 - aT)u(kT - T) + aTe(kT - T)$$

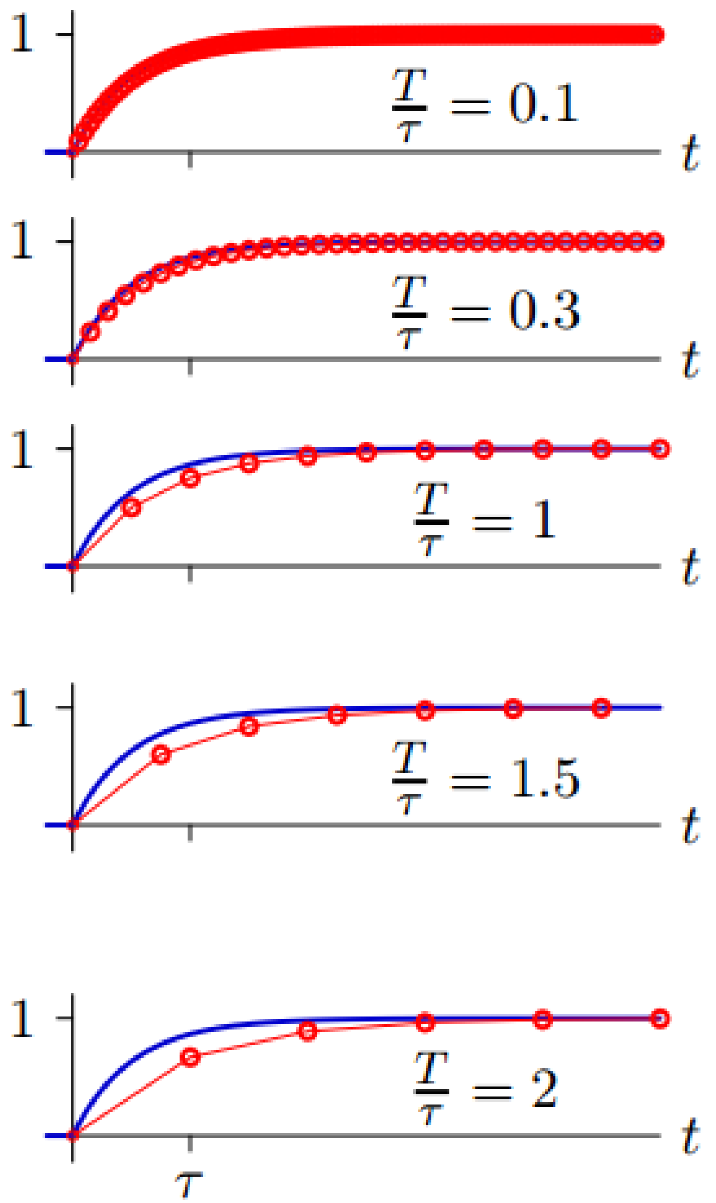
Forward Rectangular approximation is badly behaved for large aT or T/τ



$$(1 + aT)u(kT) = u(kT - T) + aTe(kT)$$

Backward Rectangular approximation is better behaved for large aT or T/τ

Since discrete stable ← continuous stable



Final Value Theorem for Z-Transform

- The final value theorem for z-transforms states that if $\lim_{k \to \infty} x(k)$ exists, then

$$\lim_{k \to \infty} x(k) = \lim_{z \to 1} (z - 1) X(z)$$

- Example: step function $U(z) = \frac{z}{z-1}$, so that

$$\lim_{k \to \infty} u(k) = \lim_{z \to 1} (z - 1) U(z) = \lim_{z \to 1} (z - 1) \frac{z}{z - 1} = \lim_{z \to 1} z = 1$$