S- and Z-Transform Table

$x(t)$	$X(s)$	$X(z)$
1. $\delta(t) = \begin{cases} 1 & t=0, \\ 0 & t=kT, k \neq 0 \end{cases}$	1	1
2. $\delta(t-kT) = \begin{cases} 1 & t=kT, \\ 0 & t \neq kT \end{cases}$	e^{-kTs}	z^{-k}
3. $u(t)$, unit step	$1/s$	$\frac{z}{z-1}$
4. t	$1/s^2$	$\frac{Tz}{(z-1)^2}$
5. t^2	$2/s^3$	$\frac{T^2 z(z+1)}{(z-1)^3}$
6. e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
7. $1-e^{-at}$	$\frac{a}{s(s+a)}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$
8. te^{-at}	$\frac{1}{(s+a)^2}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
9. t^2e^{-at}	$\frac{2}{(s+a)^3}$	$\frac{T^2 e^{-aT} z(z+e^{-aT})}{(z-e^{-aT})^3}$
10. $be^{-bt} - ae^{-at}$	$\frac{(b-a)s}{(s+a)(s+b)}$	$\frac{z[z(b-a)-(be^{-aT}-ae^{-bT})]}{(z-e^{-aT})(z-e^{-bT})}$
11. $\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$
12. $\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos \omega T)}{z^2 - 2z \cos \omega T + 1}$
13. $e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{(ze^{-aT} \sin \omega T)}{z^2 - 2ze^{-aT} \cos \omega T + e^{-2aT}}$

Solutions To Quiz 8

Q1:

If the closed loop transfer function is given as $T(s) = \frac{9K}{10s^2+6s+9K}$

What does the root locus of the closed-loop system look like?

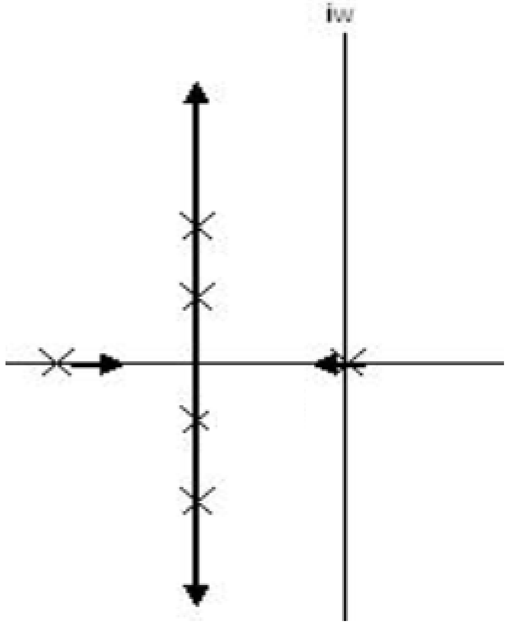
Solution:

The closedloop pole must satisfy $10s^2 + 6s + 9K = 0$

Note that the root is given as

$$s = \frac{-6 \pm \sqrt{36 - 360K}}{20}$$

This infers the root locus.



Q2

Consider the z-transform of a sequence:

$$X(z) = z^{-2}(1 + 2z^{-1})(1 - 2z^{-1})(1 + z^{-1})$$

What is the sequence $x(k)$?

Solution

$$X(z) = z^{-2}\left(1 + 2z^{-1}\right)\left(1 - 2z^{-1}\right)\left(1 + z^{-1}\right) = X(z)$$

$$X(z) = z^{-2} + z^{-3} - 4z^{-4} - 4z^{-5}$$

This infers x(k), i.e.,

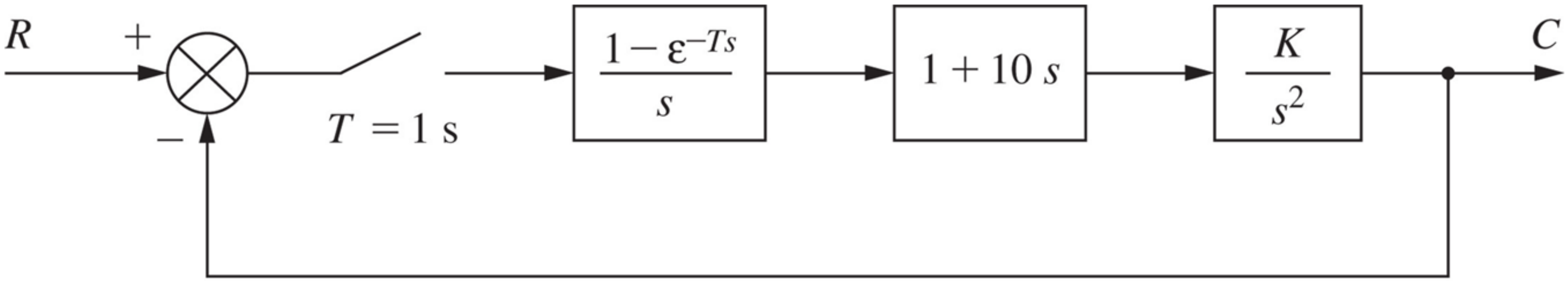
x(0)=x(1)=0,

x(2)=x(3)=1,

x(4)=x(5)=-4,

x(k)=0 for k>5

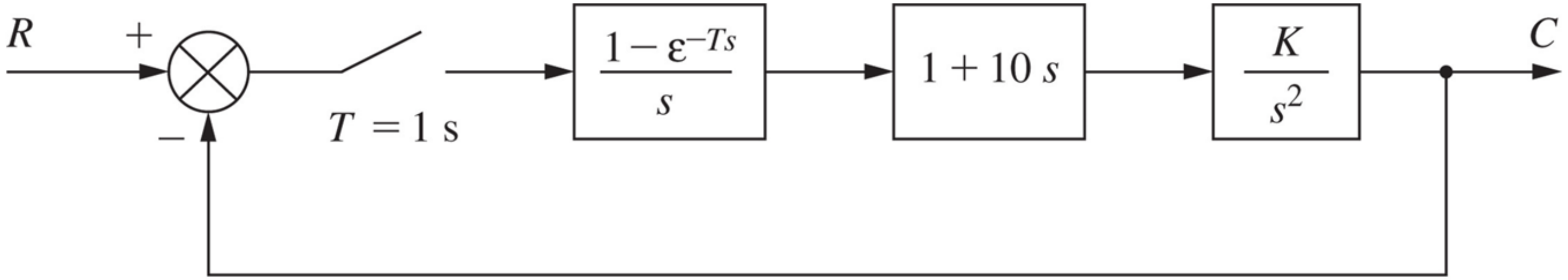
Q3



- Plot the root locus of the system in z-plane.

- Find the range of K, which makes the system stable.

Answer



- Open-loop function is

$$KG(s) = \frac{1-e^{-sT}}{s}\left[\frac{K(1+10s)}{s^2}\right]$$

- Applying the z-transform, we obtain

$$KG(z) = \frac{10.5K(z-0.9048)}{(z-1)^2}$$

- Compute the break in point:

$$\frac{dG(z)}{dz} = \frac{10.5K}{(z-1)^2} + \frac{(-2)\times 10.5K(z-0.9048)}{(z-1)^2} = 0$$

$$z = 0.81$$

$$KG(z) = \frac{z-1}{z} \quad Z\left[\frac{K(1+10s)}{s^3}\right] = K\frac{z-1}{z} \quad Z\left[\frac{1}{s^3} + \frac{10}{s^2}\right]$$

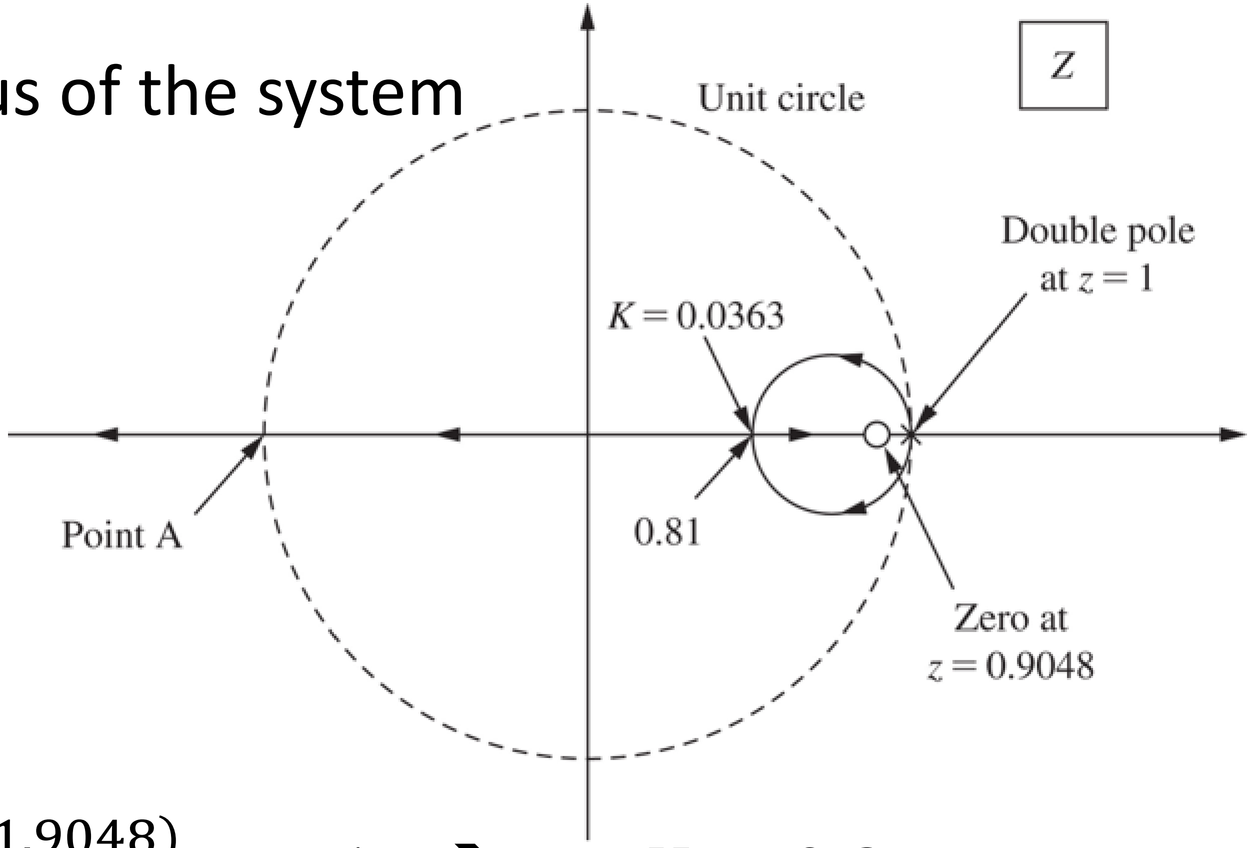
$$KG(z) = K\frac{z-1}{z}\left[\frac{\frac{1}{2}T^2 z(z+1)}{(z-1)^3} + \frac{10Tz}{(z-1)^2}\right]$$

Since T=1, we get

$$KG(z) = K\frac{z-1}{z}\left[\frac{\frac{1}{2}z(z+1)}{(z-1)^3} + \frac{10z}{(z-1)^2}\right] = K\left[\frac{\frac{1}{2}(z+1)}{(z-1)^2} + \frac{10}{z-1}\right]$$

$$KG(z) = K\frac{\frac{1}{2}(z+1) + 10(z-1)}{(z-1)^2} = K\frac{10.5z - 9.5}{(z-1)^2} = \frac{10.5K(z-0.9048)}{(z-1)^2}$$

This is the root locus of the system



At the point A, the system is marginally stable. To get the corresponding K, we need to have $1 + KG(z) = 0$ when $z = -1$.

$$\left.\frac{10.5K(z-0.9048)}{(z-1)^2}\right|_{z=-1} = \frac{10.5K(-1.9048)}{4} = -1 \quad ➔ \quad K = 0.2$$

Hence, the stability range is $0 < K < 0.2$.

Q4:

Use Z transform to solve y(k), if y(k+2)-5y(k+1)+6(k)=0, where y(0)=0 and y(1)=2.

Note:

Time Shifting Property of Z-Transform

$$x(n+k) \overset{ZT}{\longleftrightarrow} z^{k}X(z)$$
$$x(n-k) \overset{ZT}{\longleftrightarrow} z^{-k}X(z)$$

If the initial conditions are not neglected, then

$$Z[x(n+k)] = z^{k}X(z) - z^{k}\sum_{i=0}^{k-1} x(i)z^{-i}$$

$$Z[x(n-k)] = z^{-k}X(z) + z^{-k}\sum_{i=0}^{k-1} x(-i)z^{i}$$

Solve $y[k+2] - 5y[k+1] + 6y[k] = 0$, where $y[0] = 0$, $y[1] = 2$.

$$\mathcal{Z}\{y[k+2]\} - 5\mathcal{Z}\{y[k+1]\} + 6\mathcal{Z}\{y[k]\} = 0$$

Taking z transforms:

$$z^2Y(z) - zy[0] - zy[1] - 5zY(z) + 5zy[0] + 6Y(z) = 0$$

Rearranging and using initial conditions:

$$(z^2 - 5z + 6)Y(z) = 2z$$

$$Y(z) = \frac{2z}{z^2 - 5z + 6}$$

Using partial fractions:

$$Y(z) = \frac{2}{z-3} + \frac{2}{z-2}$$

Using inverse transforms straight from the table to get the solution:

$$y[k] = 2 \times 3^k - 2 \times 2^k$$