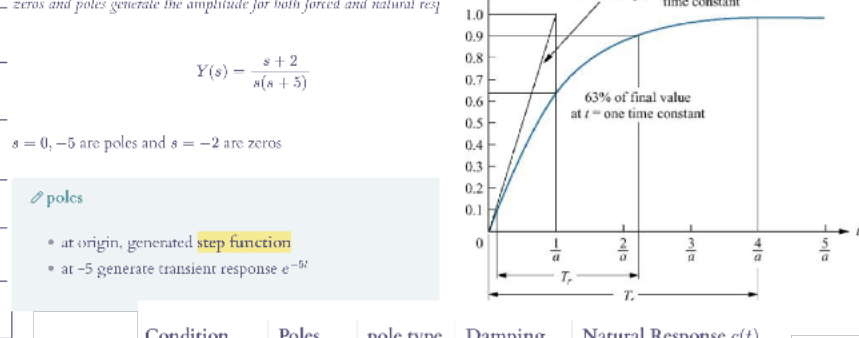


$x(t)$	$X(s)$	$X(z)$
$\delta(t) = \begin{cases} 1 & t = 0, \\ 0 & t = kT, k \neq 0 \end{cases}$	1	1
$\delta(t - kT) = \begin{cases} 1 & t = kT, \\ 0 & t \neq kT \end{cases}$	e^{-kTs}	z^{-k}
$u(t)$, unit step	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
t^2	$\frac{2}{s^3}$	$\frac{T^2 z(z+1)}{(z-1)^3}$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
$1 - e^{-at}$	$\frac{a}{s(s+a)}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$t^2 e^{-at}$	$\frac{2}{(s+a)^3}$	$\frac{T^2 e^{-aT} z(z+e^{-aT})}{(z-e^{-aT})^3}$
$be^{-bt} - ae^{-at}$	$\frac{(b-a)s}{(s+a)(s+b)}$	$\frac{z[z(b-a) - (be^{-aT} - ae^{-bT})]}{(z-e^{-aT})(z-e^{-bT})}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	$\frac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$	$\frac{(ze^{-aT} \sin \omega T)}{z^2 - 2ze^{-aT} \cos \omega T + e^{-2aT}}$

derivatives and integral
 if $\mathcal{L}\{f(t)\} = F(s)$ then we have
 $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
 $\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$
 thus the time domain output is $y(t) = 1 - e^{-at}$
 Δ time constant
 usually, $t = \frac{1}{a}$, and $y(t) = 0.63$, hence 63.2% to find the rise time.
 $w_n = \sqrt{b}$ is the frequency of oscillation of this system.

For higher derivatives we have $\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - sf^{(n-1)}(0) - f^{(n-2)}(0) - \dots - f'(0) - f(0)$
 inverse form
 $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \lim_{\omega \rightarrow \infty} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s) e^{st} ds$
 poles and zeros
 zeros and poles generate the amplitude for both forced and natural res



poles

- at origin, generated step function
- at -5 generate transient response e^{-5t}

$Y(s) = \frac{s+2}{s(s+5)}$
 $s = 0, -5$ are poles and $s = -2$ are zeros

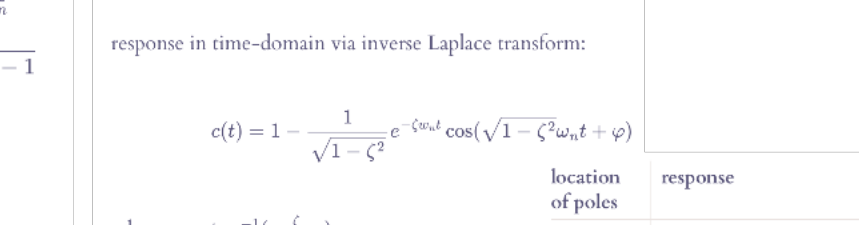
Condition	Poles	pole type	Damping Ratio (ζ)	Natural Response $c(t)$
Undamped	$\pm j\omega_n$	imaginary	$\zeta = 0$	$A \cos(\omega_n t - \varphi)$
Underdamped	$\omega_d \pm j\omega_d$	complex	$0 < \zeta < 1$	$Ae^{(-\sigma)t} \cos(\omega_d t - \varphi)$ where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$
critically damped	σ_1	real	$\zeta = 1$	$Kte^{\sigma_1 t}$
overdamped	σ_1, σ_2	real	$\zeta > 1$	$K(e^{\sigma_1 t} + e^{\sigma_2 t})$

peak time T_p
 damping ratio is defined as:
 $\zeta = \frac{\text{exponential decay frequency}}{\text{natural frequency}} = \frac{|\sigma|}{w_n}$
 $T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$
 So that $a = 2\zeta w_n$
 %OS (percent overshoot)
 $\%OS = e^{\zeta^2 / (1 - \zeta^2)} \times 100\%$

second-order systems
 general order system:
 $G(s) = \frac{b}{s^2 + as + b}$
 Thus the pole for this system:
 $s_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$
 general second order
 $G(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$
 $s_{1,2} = -\zeta w_n \pm w_n \sqrt{\zeta^2 - 1}$
 $T_p \cong \frac{4}{\zeta w_n}$

underdamped second-order step response
 Transfer function of Zero-Order hold
 Transfer function $C(s)$ is given by
 $C(s) = \frac{w_n^2}{s(s^2 + 2\zeta w_n s + w_n^2)}$
 response in time-domain via inverse Laplace transform:
 $c(t) = 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta w_n t} \cos(\sqrt{1 - \zeta^2} w_n t + \varphi)$
 where $\varphi = \tan^{-1}(\frac{\zeta}{\sqrt{1 - \zeta^2}})$

finding the discrete transfer function
 $G(s) = \frac{s^2 + 4s + 3}{s^3 + 6s^2 + 8s}$
 $G(s) = \frac{s^2 + 4s + 3}{s^3 + 6s^2 + 8s} = \frac{0.375}{s} + \frac{0.25}{s+2} + \frac{0.375}{s+4}$
 $G(t) = \mathcal{L}^{-1}\{G(s)\} = 0.375 + 0.25e^{-2t} + 0.375e^{-4t}$
 $G(z) = Z\{G(t)\} = 0.375 \frac{z}{z-1} + 0.25 \frac{z}{z-e^{-2T}} + 0.375 \frac{z}{z-e^{-4T}}$
 Let $z = e^{sT}$, we have the following definition:



stability
 system
 Stable
 Unstable
 Marginally Stable
 pole location criteria on z-plane
 All poles inside unit circle
 Any poles outside unit circle
 One or more poles on unit circle, remaining poles inside unit circle

location of poles
 Same envelope
 Same frequency
 Same overshoot
 Linearity: if $x(n) = a f_1(n) + b f_2(n)$ then $X(z) = a F_1(z) + b F_2(z)$
 Time shifting:
 $Z\{x(n-k)\} = z^{-k} X(z)$
 $Z\{x(n+k)\} = z^k X(z) - z^k \sum_{i=0}^{k-1} x(i) z^{-i}$
 $Z\{x(n-k)\} = z^{-k} X(z) + z^{-k} \sum_{i=0}^{k-1} x(-i) z^i$

resolution of A/D converter
 minimum value of the output that can be represented
 number, or $\frac{M}{2^n}$
 error = $\frac{M}{2^{n-1}}$
 where n is number of bits used for digitalisation
 A sampler is a switch that closes every T seconds:
 $r^*(t) = \sum_{k=0}^{\infty} r(kT) \delta(t - kT) \quad (t > 0)$
 Transfer function of sampled data:
 $R^*(s) = \mathcal{L}\{r^*(t)\} = \sum_{k=0}^{\infty} r(kT) e^{-ksT}$

where T is the sampling period, and $\frac{1}{T}$ is the sampling rate in cycles per second

final value theorem
 definition
 If $\lim_{k \rightarrow \infty} x(k)$ exists, then the follow exists:
 $\lim_{k \rightarrow \infty} x(k) = \lim_{z \rightarrow 1} (z-1)X(z)$

we assume $s = \alpha - j\omega$
 Location on s-plane
 Imaginary axis ($j\omega$)
 Right half-plane
 Left half-plane
 Value of α
 $\alpha = 0$
 $\alpha > 0$
 $\alpha < 0$
 Value of $e^{\alpha T}$
 $e^{\alpha T} = 1$
 $e^{\alpha T} > 1$
 $e^{\alpha T} < 1$
 Mapping on z-plane
 On unit circle
 Outside unit circle
 Inside unit circle

PID control

$$G_C(s) = K_p + \frac{K_I}{s} + K_D s$$

in time domain:

$$u(t) = K_p e(t) + K_I \int_0^t e(\eta) d\eta + K_D \frac{d(e(t))}{dt}$$

Component	Discrete-Time Equation
Proportional	$u(k) = K_p e(k)$
Integral	$u(k) = K_I T \sum_{i=1}^k e(i)$
Derivative	$u(k) = \frac{K_D}{T} [e(k) - e(k-1)]$

approximate of PID controller:

$$u(k) = K_p e(k) + K_I T \sum_{i=1}^n e(i) + \frac{K_D}{T} [e(k) - e(k-1)]$$

Assume the transfer function is given by

$$D(s) = \frac{U(s)}{E(s)} = K_0 \frac{s+a}{s+b}$$

difference equation

$$u(k) = (1 - bT)u(k-1) + K_0(aT - 1)E(k-1) + K_0 e(k)$$

The corresponding z-transform

$$\frac{U(z)}{E(z)} = \frac{K_0(aT - 1)z^{-1} + K_0}{1 + (bT - 1)z^{-1}} = \frac{K_0 z + K_0(aT - 1)}{z + (bT - 1)}$$

$$= [K_0(aT - 1) + zK_0] / [z + (bT - 1)]$$

z-transform of difference equation

example: Given $D(s) = \frac{a}{s+a}$, $u(kT) = u(k)$

$U(s)(s+a) = aE(s)$ (Laplace transform) gives
 $\frac{u(k+1) - u(k)}{T} + au(k) = ae(k)$

difference equation is $u(k+1) = (1 - aT)u(k) + aT e(k)$

z-transform is $\frac{U(z)}{E(z)} = \frac{aTz^{-1}}{1 + (aT - 1)z^{-1}} = \frac{aT}{z + (aT - 1)}$

Given the following frequency domain function: $F(s) = \frac{2s+a}{s(s-a)}$

justify whether the Final Value Theorem can or cannot be used to find the steady state value of $f(t)$.

Answer: Not applicable, since the steady state tends to infinity.

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} (3e^{at} - 1) = \infty$$

Solve $y[k+2] - 5y[k+1] + 6y[k] = 0$, where $y[0] = 0, y[1] = 2$.

$$\mathcal{Z}\{y[k+2]\} - 5\mathcal{Z}\{y[k+1]\} + 6\mathcal{Z}\{y[k]\} = 0$$

Taking z transforms:

$$z^2 Y(z) - zy[0] - zy[1] - 5zY(z) + 5zy[0] + 6Y(z) = 0$$

Rearranging and using initial conditions:

$$(z^2 - 5z + 6)Y(z) = 2z$$

$$Y(z) = \frac{2z}{z^2 - 5z + 6}$$

Using partial fractions:

$$Y(z) = \frac{2}{z-3} + \frac{2}{z-2}$$

Using inverse transforms straight from the table to get the solution:

$$y[k] = 2 \times 3^k - 2 \times 2^k$$

System Type	Transfer Function	Diagram	Characteristics
Proportional (P)	$\frac{K_p G_p}{1 + K_p G_p}$		- Affects speed of response - Cannot eliminate steady-state error
Integral (I)	$\frac{K_I}{s^2 + s + K_I}$		- Eliminates steady-state error - Output reaches 1 at steady state
PI	$\frac{K_I + sK_p}{s^2 + (1 + K_p)s + K_I}$		- P impacts response speed - I forces zero steady-state error
Derivative (D)	$\frac{K_D s}{(1 + K_D)s + 1}$		- Adds open-loop zero - Improves stability damping
Basis	$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + Z[G(s)H(s)]}$		

w/ digital sensing device	$C(z) = \frac{Z[G(s)R(s)]}{1 + Z[G(s)H(s)]}$	
w/ digital controller	$\frac{C(z)}{R(z)} = \frac{G_1(z)G_2(z)}{1 + G_1(z)Z[G_2(s)H(s)]}$	

D(s)	rule	z-transfer function D(z)	approximation	z-plane to s-plane	stability
$\frac{a}{s+a}$	forward	$\frac{a}{(z-1)/T+a}$	$s \leftarrow \frac{z-1}{T}$	$z \leftarrow sT + 1$	discrete → continuous
$\frac{a}{s+a}$	backward	$\frac{a}{(z-1)/(Tz)+a}$	$s \leftarrow \frac{z-1}{Tz}$	$z \leftarrow \frac{1}{1-Ts}$	discrete ← continuous
$\frac{a}{s-a}$	trapezoid	$\frac{a}{(2/T)((z-1)/(z+1))+a}$	$s \leftarrow \frac{2}{T} \frac{z-1}{z+1}$	$z \leftarrow \frac{1+Ts/2}{1-Ts/2}$	discrete ↔ continuous

discrete equivalent

Consider the example

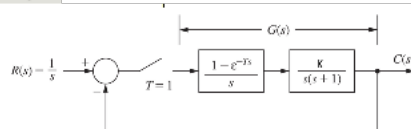
$$D(s) = \frac{U(s)}{E(s)} = \frac{a}{s+a} \rightarrow U(s)s = aE(s) - aU(s)$$

$$\rightarrow u'(t) = -au(t) + ae(t)$$

$$u(t) = \int_0^t [-au(\tau) + ae(\tau)] d\tau$$

discrete system

$$u(kT) = u(kT - T) + \int_{kT-T}^{kT} [-au(\tau) + ae(\tau)] d\tau$$



Consider the system, we have

$$G(z) = (1 - z^{-1})z \left[\frac{K}{s^2(s+1)} \right]_{(T=1)} = K \frac{z-1}{z} \left[\frac{z^2 e^{-1} + z - 2ze^{-1}}{(z-1)^2(z-e^{-1})} \right] = \frac{K(z e^{-1} + 1 - 2e^{-1})}{(z-1)(z-e^{-1})}$$

Hence, the open loop transfer function $G(z) = \frac{0.568K(z+0.718)}{(z-1)(z-0.568)}$

Steps to plot closed-loop poles

1. Derive the open loop function $K\overline{GH}$.
2. Factor numerator and denominator to get open loop zeros and poles.
3. Plot roots of $1 + K\overline{GH} = 0$ in z-Plane as K varies.

1. Loci originate on the poles of $K\overline{GH}$ and terminate on its zeros.
2. The loci are **symmetrical** with respect to the real axis.
3. The number of asymptotes is equal to the number of poles of $K\overline{GH}$, n_p , minus the number of its zeros, n_z . The angles of the asymptotes are found by $\theta_k = \frac{(2k+1)\pi}{n_p - n_z}$, $k = 0, 1, 2, \dots, (n_p - n_z - 1)$, where n_p is # of finite poles and n_z is # of finite zeros.

4. The origin of the asymptotes on the real axis is given by

$$\sigma = \frac{\sum \text{poles of } \overline{GH}(z) - \sum \text{zeros of } \overline{GH}(z)}{n_p - n_z}$$

5. The breakaway point for the locus between two poles (or the break-in point for the locus between two zeros) is found by

$$\frac{d[\overline{GH}(z)]}{dz} = 0$$

correctness: $|C(t) - Cs(t)| < \epsilon$

drift is RoC of the clock value from perfect clock. Given clock has bounded drift ρ

$$\left| \frac{dC(t)}{dt} - 1 \right| < \rho$$

	OS	RTOS
philos	time-sharing	event-driven
requirements	high-throughput	schedulability (meet all hard deadlines)
metrics	fast avg-response	ensured worst-case response
overload	fairness	meet critical deadlines

- create a child process that is identical to its parents, return 0 to child process and pid
- add a lot of overhead as duplicated. Data space is not shared

Monotonicity: $\forall t_2 > t_1 : C(t_2) > C(t_1)$

preemption & syscall

- The act of temporarily interrupting a currently scheduled task for higher priority tasks.
- NOTE: make doesn't recompile if DAG is not changed.



process

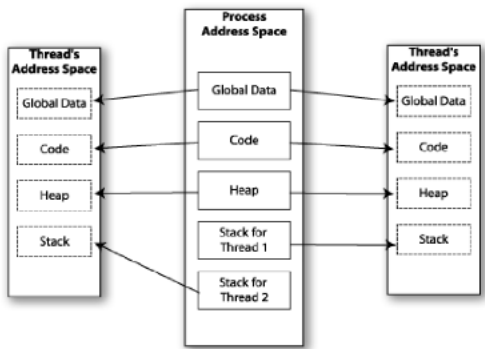
- independent execution, logical unit of work scheduled by OS
- in virtual memory:
 - Stack: store local variables and function arguments
 - Heaps: dyn located (think of malloc, calloc)
 - BSS segment: uninit data
 - Data segment: init data (global & static variables)
 - text: RO region containing program instructions

threads

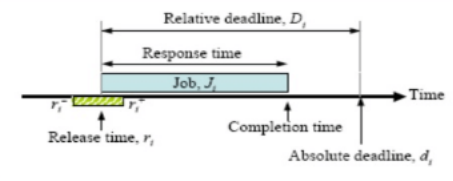
- program-wide resources: global data & instruction
- execution state of control stream
- shared address space for faster context switching
- Needs synchronisation (global variables are shared between threads)
- lack robustness (one thread can crash the whole program)

	interrupt	polling
speed	fast	slow
efficiency	good	poor
cpu-waste	low	high
multitasking	yes	yes
complexity	high	low
debug	difficult	easy

	stack	heap
creation	Member m	Member m = new Member()
lifetime	function runs to completion	delete, free is called
grow	fixed	dyn added by OS
err	stack overflow	heap fragmentation
when	size of memory is known, data size is small	large scale dyn mem

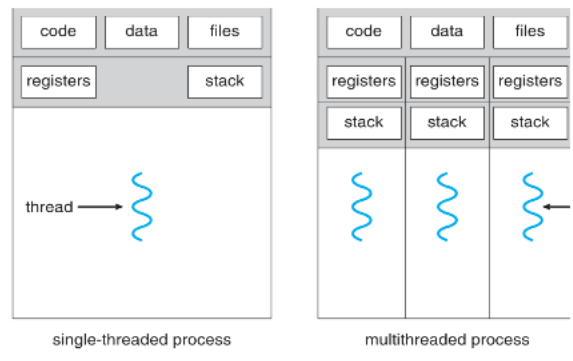
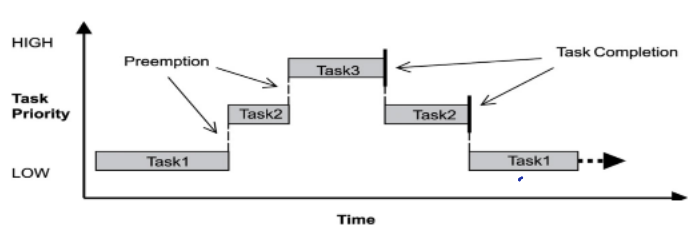


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scheduling

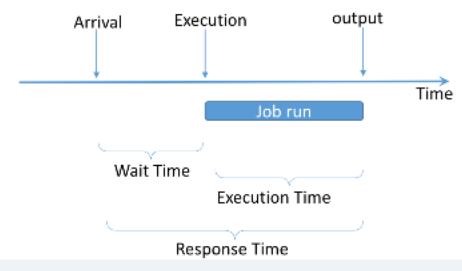
1. Priority-based preemptive scheduling



Temporal parameters:

Let the following be the scheduling parameters:

desc	var
# of tasks	n
release/arrival-time	$r_{i,j}$
absolute deadline	d_i
relative deadline	$D_i = r_{i,j} - d_i$
execution time	e_i
response time	R_i



Utilisation factor u_i

for a task T_i with execution time e_i and period p_i is given by

$$u_i = \frac{e_i}{p_i}$$

For system with n tasks overall system utilisation is $U = \sum_{i=1}^n u_i$

Period p_i of a periodic task T_i is min length of all time intervals between release times of consecutive tasks.

Phase of a Task ϕ_i is the release time $r_{i,1}$ of a task T_i , or $\phi_i = r_{i,1}$

in phase are first instances of several tasks that are released simultaneously

Representation

a periodic task T_i can be represented by:

- 4-tuple (ϕ_i, P_i, e_i, D_i)
- 3-tuple (P_i, e_i, D_i) , or $(0, P_i, e_i, D_i)$
- 2-tuple (P_i, e_i) , or $(0, P_i, e_i, P_i)$

cyclic executive

assume tasks are non-preemptive, jobs parameters with hard deadlines known.

- no race condition, no deadlock, just function call
- however, very brittle, number of frame F can be large, release times of tasks must be fixed

hyperperiod

is the least common multiple (lcm)

maximum num of arriving jobs

$$N = \sum_{i=1}^n \frac{H}{P_i}$$

Frames: each task must fit within a single frame with size $f \Rightarrow$ number of frames $F = \frac{H}{f}$

C1: A job must fit in a frame, or $f \geq \max e_i \forall 1 \leq i \leq n$ for all tasks

C2: hyperperiod has an integer number of frames, or $\frac{H}{T} = \text{integer}$

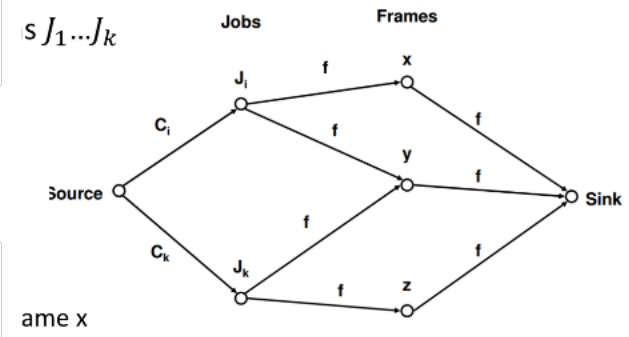
deadline-monotonic

C3: $2f - \gcd(P_i, f) \leq D_i$ per task.

- if every task has period equal to relative deadline, same as RM
- arbitrary deadlines then DM performs better than RM
- RM always fails if DM fails

Flow Graph for hyper-period

- Denote all jobs in hyperperiod of F frames as $J_1 \dots J_F$
- Vertices:
 - k job vertices J_1, J_2, \dots, J_k
 - F frame vertices x, y, \dots, z
- Edges:
 - (source, J_i) with capacity $C_i = e_i$
 - Encode jobs' compute requirements
 - (J_i, x) with capacity f iff J_i can be scheduled in frame x
 - encode periods and deadlines
 - edge connected job node and frame node iff
 1. job arrives **before** or at the starting time
 2. job's absolute deadline **larger** or equal to the ending time
 - (f, sink) with capacity f
 - encodes limited computational capacity in frame x



Priority Inheritance Protocol (PIP)

idea: increase the priorities only upon resource contention

avoid NPCS drawback

would still run into deadlock (think of RR task resource access)

idea: find k such that $\sum_{k=1}^k \frac{e_i}{p_i} \leq 1$

general solution for RM-schedulability

The time demand function for task i ; $1 \leq i \leq n$:

$$w_i(t) = \sum_{k=1}^k \frac{t}{p_i} \cdot e_k \leq t$$

$$\therefore 0 \leq t \leq p_i$$

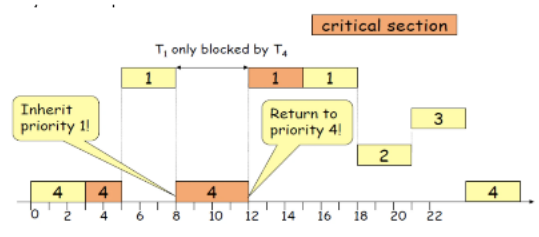
holds a time instant t chosen as $t = k \cdot p_j$, ($j = 1, \dots, n$) and $k_j = 1, \dots, \lfloor \frac{t}{p_j} \rfloor$

Priority Inheritance Protocol (PIP)

idea: increase the priorities only upon resource contention

avoid NPCS drawback

would still run into deadlock (think of RR task resource access)



rules:

- When a task T1 is blocked due to non availability of a resource that it needs, the task T2 that holds the resource and consequently blocks T1, and T2 inherits the current priority of task T1.
- T2 executes at the inherited priority until it releases R.
- Upon the release of R, the priority of T2 returns to the priority it held when it acquired the resource

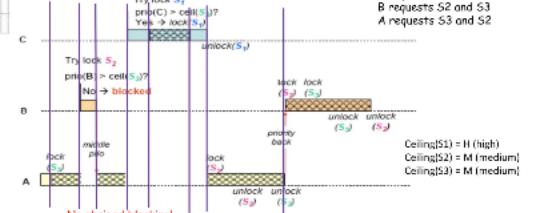
Priority Ceiling Protocol (PCP)

idea: extends PIP to prevent deadlocks

- If lower priority task TL blocks a higher priority task TH, priority(TL) ← priority(TH)
- When TL releases a resource, it returns to its normal priority if it doesn't block any task. Or it returns to the highest priority of the tasks waiting for a resource held by TL
- Transitive
 - T1 blocked by T2: priority(T2) ← priority(T1)
 - T2 blocked by T3: priority(T3) ← priority(T1)

priority ceiling(R): highest priority. Each resource has fixed priority ceiling

Assume C has a high priority (H), A has a low priority (L), and B has a medium priority (M). C requests S1, B requests S2 and S3, A requests S3 and S2



rate-monotonic

running on uniprocessor, tasks are preemptive, no OS overhead for preemption

task T_i has higher priority than task T_j if $p_i < p_j$

schedulability test for RM (Test 1)

Given n periodic processes, independent and preemptable, $D_i \geq p_i$ for all processes, periods of all tasks are integer multiples of each other

a sufficient condition for tasks to be scheduled on uniprocessor: $U = \sum_{i=1}^n \frac{e_i}{p_i} \leq 1$

schedulability test for RM (Test 2)

a sufficient but not necessary condition is $U \leq n \cdot (2^{\frac{1}{n}} - 1)$ for n periodic tasks

for $n \rightarrow \infty$, we have $U < \ln(2) \approx 0.693$

earliest-deadline first (EDF)

depends on closeness of absolute deadlines

EDF schedulability test 1

set of n periodic tasks, each whose relative deadline is equal to or greater than its period
 iff $\sum_{i=1}^n (\frac{e_i}{p_i}) \leq 1$

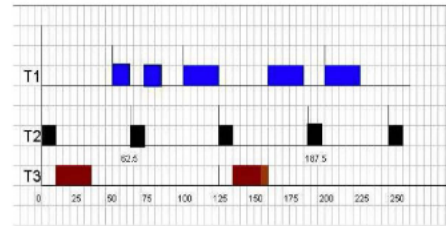
EDF schedulability test 2

relative deadlines are not equal to or greater than their periods

$$\sum_{i=1}^n \left(\frac{e_i}{\min(D_i, p_i)} \right) \leq 1$$

$$G(z) = \frac{U(z)}{E(z)} = \frac{(z+1)(z-0.9512)}{(z-0.9039)(z-0.8616)}$$

Using DM to schedule T1(50; 50; 25; 100); T2(0; 62.5; 10; 20); T3(0; 125; 25; 50)



So $U(z)(1 + 1.7655z^{-1} - 0.7788z^{-2}) = E(z)(1 + 0.0488z^{-1} - 0.9512z^{-2})$

By applying inverse Z-Transform to the above equation, the result is:

$$u(k) = e(k) + 0.0488e(k-1) - 0.9512e(k-2) - 1.7655u(k-1) + 0.7788u(k-2)$$