Q1. $b = \bar{y} - a\bar{x}$

$ax + b$

$(x_1, y_1) \cdots (x_N, y_N)$

$\bar{x} = 0 \Rightarrow b = \bar{y}$

Q2. $X \in \mathbb{R}^{N \times d}$

$X^T X \in \mathbb{R}^{d \times d}$

$\operatorname{rank}(X^T X) \leq \min\{\operatorname{rank}(X^T), \operatorname{rank}(X)\} = \min\{N, d\}$

$\operatorname{rank}(X^T X) < d$

$\Rightarrow N < d \Rightarrow \operatorname{rank}(X) < d$

Q3. $\underline{\underline{b}}$

Q4. $\underline{\underline{a}}$

Q5. $O(Nd)$

Q6. $\underline{\underline{a}}$

Q7. $\underline{\underline{b}}$ (NP)

Q8. $T = \{t_1, t_2, \ldots t_N\}$

$$\lambda^{ML} = \underset{\lambda}{\arg\max}\ P(T \mid \lambda) \rightarrow \prod \Pr(t_i \mid \lambda)$$

$$= \underset{\lambda}{\arg\min} - \sum_{i=1}^{N} \log P(t_i \mid \lambda)$$

$$= \underset{\lambda}{\arg\min} - \sum_{i=1}^{N} \log \lambda e^{-\lambda t_i}$$

$\log \lambda + \log e^{-\lambda t_i}$

$\rightarrow - \sum_{i=1}^{N} \log \lambda \rightarrow N \log \lambda$

$- \sum_{i=1}^{N} \log e^{-\lambda t_i}$

$\sum \lambda t_i$

$$= \underset{\lambda}{\operatorname{argmin}} \; - N \log \lambda + \sum \lambda t_i$$

$$= \underset{\lambda}{\operatorname{argmin}} \; - N \log \lambda + \lambda \sum t_i$$

$$\frac{\partial L}{\partial \lambda} = -\frac{N}{\lambda} + \sum t_i = 0$$

$$\Rightarrow \quad \lambda = \frac{N}{\sum t_i} = \frac{1}{\bar{t}}$$

$\lambda e^{-\lambda x}$

$\downarrow$

$\lambda = \frac{1}{E[X]}$

Q9.

$$f(x) = \begin{cases} 0 & x \leq 1 \\ 1 & x > 1 \end{cases}$$

$$a^{LS} = \underset{a}{\operatorname{argmin}} \; \underset{x \sim P_X}{E} \left(f(x) - ax \right)^2$$

$$P_X(x) = \begin{cases} \frac{1}{2} & x \in [0, 2] \\ 0 & \text{otherwise} \end{cases}$$

$[a, b]$

$\frac{1}{b-a}$

$$a^{LS} = \arg\min_a \int_{-\infty}^{\infty} P_X(x) \left(f(x) - ax\right)^2 dx$$

$$= \arg\min_a \int_0^2 \frac{1}{2} \left(f(x) - ax\right)^2 dx$$

$$= \arg\min_a \int_0^1 \frac{1}{2} \left(0 - ax\right)^2 dx + \int_1^2 \frac{1}{2} \left(1 - ax\right)^2 dx$$

$$= \arg\min_a \frac{1}{2} \int_0^1 a^2 x^2 \, dx + \frac{1}{2} \int_1^2 1 + a^2 x^2 - 2ax \, dx$$

$$= \arg\min_a \frac{1}{2} \left(\frac{a^2}{3} x^3 \right) \Big|_0^1 + \frac{1}{2} \left(x + \frac{a^2}{3} x^3 - ax^2 \right) \Big|_1^2$$

$$= \arg\min_a \frac{a^2}{6} + \frac{1}{2} \left(2 + \frac{8a^2}{3} - 4a - 1 - \frac{a^2}{3} + a \right)$$

$$= \underset{a}{\arg\min} \quad \frac{a^2}{6} + \frac{1}{2}\left(1 + \frac{7a^2}{3} - 3a\right)$$

$$= \underset{a}{\arg\min} \quad \frac{4a^2}{3} + \frac{1}{2} - \frac{3}{2}a$$

$$\frac{\partial L}{\partial a} = \frac{8a}{3} - \frac{3}{2} = 0 \implies a^{LS} = \frac{9}{16}$$

Q9.b

$$\underset{x \sim P_X}{E}\left(f(x) - a^{LS} x\right)^2$$

$$= \frac{4(a^{LS})^2}{3} + \frac{1}{2} - \frac{3}{2}a^{LS}$$

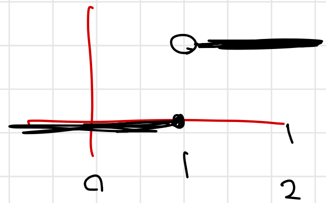
$$= \frac{4}{3}\left(\frac{9}{16}\right)^2 + \frac{1}{2} - \frac{3}{2}\left(\frac{9}{16}\right)$$

$$= \frac{27}{64} - \frac{27}{32} + \frac{1}{2} = \frac{27 - 2\times 27}{64} + \frac{1}{2}$$

$$= \frac{1}{2} - \frac{27}{64}$$

Q9.c

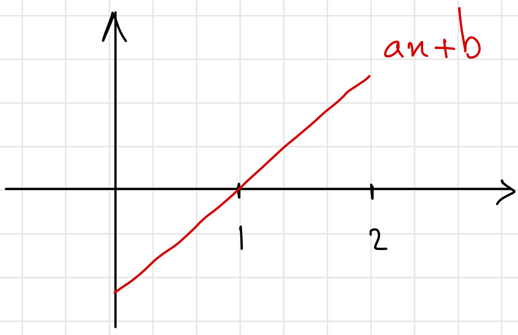
$$\hat{y} = \text{sign}(ax+b)$$



$$\mathop{E}_{x \sim P_X} \ell^{0-1}\left(f(x), \text{sign}(ax+b)\right)$$

$$= \int_0^2 \frac{1}{2} \ell^{0-1}\left(f(x), \text{sign}(ax+b)\right) dx$$

$$= \frac{1}{2}\int_0^1 \ell^{0-1}\left(0, \text{sign}(ax+b)\right) dx + \frac{1}{2}\int_1^2 \ell^{0-1}\left(1, \text{sign}(ax+b)\right) dx$$



$\Rightarrow L = 0$

$(1, 0) \rightarrow \quad ax+b \Rightarrow a+b=0$

$a > 0$