COMPSCI 4ML3: Tutorial 1

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Introduction

Linear algebra is useful to operate on sets of linear equations.

- Example: The set of linear equations

$$x_1 + x_2 + x_3 = 5$$
$$x_1 - 2x_2 - 3x_3 = -1$$
$$2x_1 + x_2 - x_3 = 3$$

Can be written in matrix format as $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$$

Notations I

$A \in \mathbb{R}^{m \times n}$: Matrix with m rows and n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

A_{ij} denotes the entry in row i and column j of matrix A

Notations II

$x \in \mathbb{R}^n$: Vector with n-dimensions

$$\text{Column vector: } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ Row vector: } x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$$

x_i denotes the ith element of vector

Definitions I

- Main diagonal of matrix: Entries a_{ij} where $i = j$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Definitions II

- Diagonal matrix: Every entry except the main diagonal is zero

$$A = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}$$

  Also denoted by $\text{Diag}(a_1, \dots, a_n)$

- Trace: Sum of the entries in main diagonal

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$$

Definitions III

- Identity matrix: $I = \text{Diag}(1, \ldots, 1)$

$$I = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

- Transpose: If $A \in \mathbb{R}^{m \times n}$ then $A^T \in \mathbb{R}^{n \times m}$, where $(A^T)_{ij} = A_{ji}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \ A^T = \begin{bmatrix} a_{11} & a_{21} & \ldots & a_{m1} \\ a_{12} & a_{22} & \ldots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{mn} \end{bmatrix}$$

Inner Product

Given $x, y \in \mathbb{R}^n$, the product $x^T y \in \mathbb{R}$ is called the **inner product** or **dot product**

$$x^T y = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i$$

Also denoted by $\langle x, y \rangle$

Matrix Multiplication

Given matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ the product $C = AB \in \mathbb{R}^{m \times p}$ is given by

$$AB = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{n} a_{1i} b_{i1} & \sum_{i=1}^{n} a_{1i} b_{i2} & \dots & \sum_{i=1}^{n} a_{1i} b_{ip} \\ \sum_{i=1}^{n} a_{2i} b_{i1} & \sum_{i=1}^{n} a_{2i} b_{i2} & \dots & \sum_{i=1}^{n} a_{2i} b_{ip} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} a_{mi} b_{i1} & \sum_{i=1}^{n} a_{mi} b_{i2} & \dots & \sum_{i=1}^{n} a_{mi} b_{ip} \end{bmatrix}$$

Matrix Multiplication: Properties

- Compatibility: Matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ are compatible iff $n = p$, which means they can be multiplied
- Matrix multiplication is associative: $(AB)C = A(BC)$
- Matrix multiplication is distributive: $A(B + C) = AB + AC$
- Matrix multiplication is not commutative: $AB \neq BA$

Matrix-Vector Multiplication

Given a matrix $A \in \mathbb{R}^{m \times n}$ and a column vector $x \in \mathbb{R}^n$ the product $Ax \in \mathbb{R}^m$ is defined as follows

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i}x_i \\ \sum_{i=1}^n a_{2i}x_i \\ \vdots \\ \sum_{i=1}^n a_{mi}x_i \end{bmatrix}$$

Can be interpreted as a **linear combination** of columns

$$Ax = \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix} x_1 + \begin{bmatrix} | \\ a_2 \\ | \end{bmatrix} x_2 + \dots + \begin{bmatrix} | \\ a_n \\ | \end{bmatrix} x_n$$

Matrix-Matrix Multiplication I

AB can be computed by inner product of rows of A and columns of B

$$AB = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \dots & b_p \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \dots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & \dots & a_2^T b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b_1 & a_m^T b_2 & \dots & a_m^T b_p \end{bmatrix}$$

Matrix-Matrix Multiplication II

- AB can also be interpreted by vector-matrix multiplication of A and columns of B

$$AB = A \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \dots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \dots & Ab_p \\ | & | & & | \end{bmatrix}$$

- Multiplication by identity: If $A \in \mathbb{R}^{m \times n}$, $AI_n = I_m A = A$

Inverse of a Matrix

The inverse of a square matrix $A \in \mathbb{R}^{n\times n}$ is a **unique** matrix denoted as $A^{-1} \in \mathbb{R}^{n\times n}$ such that

$$A^{-1}A = I = AA^{-1}$$

Not every matrix has an inverse:

- If the inverse exists, the matrix A is called **invertible** or **non-singular**
- If the inverse does not exist, the matrix A is called **non-invertible** or **singular**

Transpose: Properties

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A + B)^T = A^T + B^T$

Inverse: Properties

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^{T} = (A^{T})^{-1}$

Norms

A function $f : \mathbb{R}^n \to \mathbb{R}$ is a norm if it satisfies:

1. Non-negativity: $\forall x \in \mathbb{R}^n,\ f(x) \geq 0$
2. Definiteness: $f(x) = 0$ iff $x = 0$
3. Homogeneity: $\forall x \in \mathbb{R}^n,\ t \in \mathbb{R},\ f(tx) \leq |t| f(x)$
4. Triangle inequality: $\forall x, y \in \mathbb{R}^n,\ f(x+y) \leq f(x) + f(y)$

Vector Norms

- Euclidean norm(ℓ_2 norm): $\|x\|_2 = \sqrt{\sum_{i=0}^{n} x_i^2}$

  $\|x\|_2^2 = x^T x$
- ℓ_1 norm: $\|x\|_1 = \sum_{i=0}^{n} |x_i|$
- ℓ_∞ norm: $\|x\|_\infty = \max_i |x_i|$
- ℓ_p norm: $\|x\|_p = \left(\sum_{i=0}^{n} |x_i|^p \right)^{1/p}$

Linear Dependency

Given a set of vectors $\{x_1, \ldots, x_n\} \subset \mathbb{R}^m$ they are

- **Linearly independent** if no vector can be represented as a linear combination of the remaining vectors:

$$\forall i \in [n], \forall \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n\} \subset \mathbb{R}, \; x_i \neq \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j x_j$$

- **Linearly dependent** if one of the vectors can be represented as a linear combination of the remaining vectors:

$$\exists i \in [n], \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n\} \subset \mathbb{R}, \; x_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j x_j$$

Linear Dependency: Example

The following vectors are linearly dependent

$$x_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}, x_2 = \begin{bmatrix}4\\1\\5\end{bmatrix}, x_3 = \begin{bmatrix}2\\-3\\-1\end{bmatrix}$$

since $x_3 = -2x_1 + x_2$

Span

Given a set of vectors $S = \{x_1, \ldots, x_n\}$, the span of S is the set of vectors that can be written as the linear combination of vectors in set S

$$\text{span}(\{x_1, \ldots, x_n\}) = \{y : y = \sum_{i=1}^{n} \alpha_i x_i,\ \alpha_i \in \mathbb{R}\}$$

If $x_1, \ldots, x_n \in \mathbb{R}^n$ are linearly independent

$$\text{span}(\{x_1, \ldots, x_n\}) = \mathbb{R}^n$$

Rank

For a matrix $A \in \mathbb{R}^{m \times n}$

- The **column rank** of a matrix A is the maximum number of linearly independent columns of A
- The **row rank** of a matrix A is the maximum number of linearly independent rows of A
- The column rank and row rank are equal and they are called the **rank** of matrix A
  - If $\text{rank}(A) \leq m$, the rows are linearly dependent
  - If $\text{rank}(A) \leq n$, the columns are linearly dependent

Rank: Properties

For $A \in \mathbb{R}^{m \times n}$

- $\text{rank}(A) \leq \min(m, n)$ and if $\text{rank}(A) = \min(m, n)$, A is called **full rank**
- $\text{rank}(A) = \text{rank}(A^T)$
- If $B \in \mathbb{R}^{n \times p}$, $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$
- If $B \in \mathbb{R}^{m \times n}$, $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$

Solving Linear System of Equations I

A set of m equations with n variables $x_1, \ldots, x_n$ can be represented by matrices

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

Solving Linear System of Equations II

We can represent the equations in previous slide in matrix form $Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Note It can be interpreted as linear combination of columns of A

$$\begin{bmatrix} | \\ a_1 \\ | \end{bmatrix} x_1 + \begin{bmatrix} | \\ a_2 \\ | \end{bmatrix} x_2 + \dots + \begin{bmatrix} | \\ a_n \\ | \end{bmatrix} x_n = \begin{bmatrix} | \\ b \\ | \end{bmatrix}$$

Solving Linear System of Equations: Solution Set

If $A \in \mathbb{R}^n$ is invertible, there exists a unique solution

$$x = A^{-1}b$$

In terms of rank for the system $Ax = b$ we have

- If $\text{rank}(A) = \text{rank}[A|b]$ we know that $b \in \mathcal{R}(\mathcal{A})$
  - If $\text{rank}(A) = \text{rank}[A|b] = n$, the system has a **unique** solution
  - If $\text{rank}(A) = \text{rank}[A|b] < n$, the system has **infinitely many** solutions
- If $\text{rank}(A) < \text{rank}[A|b]$, we know that $b \notin \mathcal{R}(\mathcal{A})$ and the system is inconsistent and has **no solution**

Additional Material and Examples

Trace: Properties

- $\text{tr}(A) = \text{tr}(A^T)$
- $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$
- $\forall t \in \mathbb{R}$, $\text{tr}(tA) = t\text{tr}(A)$
- if AB is square, $\text{tr}(AB) = \text{tr}(BA)$
- if ABC is square, $\text{tr}(ABC) = \text{tr}(CAB) = \text{tr}(BCA)$

Norm Inequalities

- Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$
- Cauchy–Schwarz inequality: For vectors u and v, $|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle$ or $|\langle u, v\rangle| \leq \|u\|\|v\|$

Range and Projection

- The range of a matrix $A \in \mathbb{R}^{m\times n}$ is denoted by $\mathcal{R}(A)$ and is the span of columns of A

$$\mathcal{R}(A) = \{y \in \mathbb{R}^m : y = Ax,\ x \in \mathbb{R}^m\}$$

- The projection of a vector $y \in \mathbb{R}^m$ onto $\text{span}(\{x_1, \ldots, x_n\})$, $x_i \in \mathbb{R}^m$ is a vector in the span that is as close as possible to y with respect to ℓ_2 norm

$$\text{Proj}(y; \{x_1, \ldots, x_n\}) = \underset{v \in \text{span}(\{x_1,\ldots,x_n\})}{\arg\min} \|y - v\|_2$$

Nullspace

Nullspace of matrix $A \in \mathbb{R}^{m \times n}$ is the set of all vectors that their matrix vector multiplication by A is equal to 0

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$$

Outer Product

Given vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, the **outer product** $xy^T \in \mathbb{R}^{n \times m}$ is defined as

$$xy^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \dots & y_m \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \dots & x_1 y_m \\ x_2 y_1 & x_2 y_2 & \dots & x_2 y_m \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1 & x_n y_2 & \dots & x_n y_m \end{bmatrix}$$

Matrix Norms

- $\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$
  For $A \in \mathbb{R}^{m \times n}$
- $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|$, which is the maximum of absolute column sum
- $\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$, which is the maximum of absolute row sum
- Frobenius norm: $\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|} = \sqrt{tr(A^T A)}$

Projection and Linear equations

If $A \in \mathbb{R}^{m\times n}$ is full rank and $n \leq m$ for $y \in \mathbb{R}^m$

$$\text{Proj}(y; A) = \arg\min_{v\in\mathcal{R}(A)} \|y - v\|_2 = A(A^TA)^{-1}A^Ty$$

Remember If $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$, $\hat{y} = Ax$ is a vector in $\mathbb{R}^m$ and given $y \in \mathbb{R}^m$

$$\|\hat{y} - y\|_2^2 = \sum_{i=1}^{m}(\hat{y}_i - y_i)^2 = (\hat{y} - y)^T(\hat{y} - y) = (Ax - y)^T(Ax - y)$$

Range and Nullspace

For $A \in \mathbb{R}^{m \times n}$,$\mathcal{R}(A^T)$ and $\mathcal{N}(A)$ are disjoint sets that span the entire space of $\mathbb{R}^n$. These are called **orthogonal complements** and are denoted as $\mathcal{R}(A^T) = \mathcal{N}(A)^{\perp}$

$$\{u + v : u \in \mathcal{R}(A^T),\ v \in \mathcal{N}(A)\} = \mathbb{R}^n,\ \mathcal{R}(A^T) \cap \mathcal{N}(A) = \{0\}$$

More on Linear System of Equations

For the linear equation $Ax = b$, where $A \in \mathbb{R}^{n \times n}$, the following are equivalent

- $Ax = b$ has a unique solution in $\mathbb{R}^n$
- A is invertible
- $\text{rank}(A) = n$
- $Ax = 0$ has a unique solution of $x = 0$
- $\mathcal{N}(A) = \{0\}$

Determinant I

The determinant of $A \in \mathbb{R}^{2\times 2}$ can be computed as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

The determinant of $A \in \mathbb{R}^{3\times 3}$ can be computed as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_32 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$-a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Determinant II

- **Minors:** For a square matrix $A \in \mathbb{R}^{n \times n}$, the minor of the entry a_{ij} denoted by M_{ij} is the determinant of a smaller square matrix by removing the ith row and jth column of A
- **Cofactors:** For a square matrix $A \in \mathbb{R}^{n \times n}$, the cofactor of the entry a_{ij} denoted by C_{ij} is expressed as $(-1)^{i+j} M_{ij}$
- The determinant of the square matrix $A \in \mathbb{R}^{n \times n}$ can be computed by cofactor expansion along column j or row i

$$\text{Column expansion: } \det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} M_{ij}$$

$$\text{Row expansion: } \det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{ij}$$

Determinant: Properties

For a square matrix $A \in \mathbb{R}^n$

- $|A| = |A^T|$
- $|AB| = |A||B|$
- $|A| = 0$ iff A is singular
- If A is non-singular, $|A| = \dfrac{1}{|A^{-1}|}$

Inverse: Calculation

- **Cofactor Matrix:** The matrix consisted of cofactors

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

- **Adjoint Matrix**: The adjoint of the square matrix A is the transpose of cofactor matrix: $\text{adj}(A) = C^T$
- The inverse of a square matrix $A \in \mathbb{R}^n$ can be calculated as

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$$

Solving Linear System of Equations I

For a linear equation $Ax = b$,

$$Ax = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Denote D_i as the matrix that replaces the column i in matrix A with the column vector b

$$D_i = \begin{bmatrix} | & | & & | & | & | & & | \\ a_1 & a_2 & \dots & a_{i-1} & b & a_{i+1} & \dots & a_n \\ | & | & & | & | & | & & | \end{bmatrix}$$

Solving Linear System of Equations II

The solution set of $Ax = b$ can be found by **Cramer's rule**

$$\forall i \in [n],\ x_i = \frac{\det(D_i)}{\det(A)}$$

First Example Revisited I

Consider the linear system of equations $Ax = b$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$$

D_is are constructed as

$$D_1 = \begin{bmatrix} 5 & 1 & 1 \\ -1 & -2 & -3 \\ 3 & 1 & -1 \end{bmatrix}, D_2 = \begin{bmatrix} 1 & 5 & 1 \\ 1 & -1 & -3 \\ 2 & 3 & -1 \end{bmatrix}, D_3 = \begin{bmatrix} 1 & 1 & 5 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

First Example Revisited II

The determinant are computed

$$|A| = 5,$$
$$|D_1| = 20, |D_2| = -10, |D_3| = 15$$

Therefore,

$$x_1 = \frac{20}{5} = 4,\ x_1 = \frac{-10}{5} = -2,\ x_1 = \frac{15}{5} = 3$$