COMPSCI 4ML3: Tutorial 2

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Symmetric Matrix

A square matrix $A \in \mathbb{R}^{n \times n}$ is

- Symmetric if $A = A^T$. We say $A \in \mathbb{S}^n$.
- Anti-symmetric if $A = -A^T$

Given any square matrix $A \in \mathbb{R}^{n \times n}$

- $A + A^T$ is symmetric
- $A - A^T$ is anti-symmetric

A square matrix can be written as the sum of a symmetric and an anti-symmetric matrix

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

Quadratic Forms I

Given a square matrix $A \in \mathbb{R}^{n \times n}$ and a vector $x \in \mathbb{R}^n$ the **scalar** $x^T A x \in \mathbb{R}$ is called a quadratic form

$$x^T A x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} \sum_{i=1}^n a_{1i} x_i \\ \sum_{i=1}^n a_{2i} x_i \\ \vdots \\ \sum_{i=1}^n a_{ni} x_i \end{bmatrix} = \sum_{j=1}^n \left(x_j \sum_{i=1}^n a_{ji} x_i \right) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$$

Quadratic Forms Example

Write $f(x) = 2x_3^2 + x_1^2 + 3x_1x_2$ as a quadratic form. $(x \in \mathbb{R}^3)$

Quadratic Forms Example

Write $f(x) = 2x_3^2 + x_1^2 + 3x_1x_2$ as a quadratic form. ($x \in \mathbb{R}^3$)

$$f(x) = x^T \begin{bmatrix} 1 & 1.5 & 0 \\ 1.5 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} x$$

Positive Definite Matrix

Given a symmetric matrix $A \in \mathbb{S}^n$

- A is **positive definite(PD)** if $x^T A x > 0$ for all non-zero vectors $x \in \mathbb{R}^n$. Also denoted as $A \succ 0$. The set of all positive definite matrices is denoted as $\mathbb{S}^n_{++}$
- A is **positive semidefinite(PSD)** if $x^T A x \geq 0$ for all vectors $x \in \mathbb{R}^n$. Also denoted as $A \succeq 0$. The set of all positive semidefinite matrices is denoted as $\mathbb{S}^n_{+}$

Negative Definite Matrix

Given a symmetric matrix $A \in \mathbb{S}^n$

- A is **negative definite(ND)** if $x^T Ax < 0$ for all non-zero vectors $x \in \mathbb{R}^n$. Also denoted as $A \prec 0$.
- A is **negative semidefinite(NSD)** if $x^T Ax \leq 0$ for all vectors $x \in \mathbb{R}^n$. Also denoted as $A \preceq 0$.

A symmetric matrix $A \in \mathbb{S}^n$ is **indefinite** if it is neither positive semidefinite nor negative semidefinite

$$\exists\, x_1, x_2 \in \mathbb{R}^n,\ x_1^T Ax_1 > 0,\ x_2^T Ax_2 < 0$$

Positive and Negative Definite Matrices

Given a symmetric matrix $A \in \mathbb{S}^n$, the matrix $-A \in \mathbb{S}^n$ is

- negative definite if A is positive definite
- positive definite if A is negative definite

A positive or negative definite matrix is always *full rank* and *invertible*.

Positive and Negative Definite Matrices

Example Given **any** matrix $A \in \mathbb{R}^{m \times n}$, the matrix $G = A^T A$ is positive semidefinite, which is called **Gram** matrix.
proof.

Positive and Negative Definite Matrices

Example Given **any** matrix $A \in \mathbb{R}^{m \times n}$, the matrix $G = A^T A$ is positive semidefinite, which is called **Gram** matrix.
proof.

$$\forall x \in \mathbb{R}^d \quad x^T G x = x^T A^T A x = (Ax)^T (Ax) = ||Ax||_2^2 \geq 0$$

Eigenvectors and Eigenvalues

Given a square matrix $A \in \mathbb{R}^{n \times n}$, the non-zero vector $x \in \mathbb{C}^n$ is called the eigenvector of A and $\lambda \in \mathbb{C}$ is called the corresponding eigenvalue if

$$Ax = \lambda x$$

Multiplying A by its eigenvector x results in a vector in the same direction as x, scaled by the corresponding eigenvalue λ

Finding Eigenvectors and Eigenvalues

Rewriting $Ax = \lambda x$ results in

$$(A - \lambda I)x = 0$$

There exists a non-zero eigenvector iff the nullspace of $(A - \lambda I)$ is non-empty, which implies $(A - \lambda I)$ is *singular*

$$|(A - \lambda I)| = 0$$

Expanding the determinant results in a polynomial of degree at most n

- Eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are the roots of the polynomial
- Eigenvectors can be determined by solving linear equations $(A - \lambda_i I)x_i = 0$

Finding Eigenvectors and Eigenvalues

Example. Find the eigenvalues of $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$.

Eigenvalues: Properties I

Given a square matrix $A \in \mathbb{R}^n$

- The trace of A is equal to the sum of its eigenvalues

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$$

- The determinant of A is equal to the product of its eigenvalues

$$|A| = \prod_{i=1}^{n} \lambda_i$$

- Rank of A is equal to the number of its non-zero eigenvalues

Eigenvalues: Properties II

- If A is invertible, x_i are also eigenvectors of A^{-1} with corresponding eigenvalues $(1/\lambda_i)$, i.e., $A^{-1}x_i = (1/\lambda_i)(x_i)$
- Eigenvalues of $D = \text{Diag}(d_1, \ldots, d_n)$ are $d_1, \ldots, d_n$

$$|D - \lambda I| = \prod_{i=1}^{n} (d_i - \lambda)$$

Orthogonal Matrix

- Two vectors $x, y \in \mathbb{R}^n$ are orthogonal if $x^T y = 0$
- Vector $x \in \mathbb{R}^n$ is normalized if $\|x\|_2 = 1$
- A matrix $U \in \mathbb{R}^{n \times n}$ is orthogonal if its columns are orthogonal and are normalized(orthonormal)

$$U^T U = I = UU^T$$

  and $U^{-1} = U^T$
- When multiplied to a vector $x \in \mathbb{R}^n$, the orthogonal matrix $U \in \mathbb{R}^{n \times n}$ will not change the Euclidian norm

$$\|Ux\|_2 = \|x\|_2$$

Eigenvectors and Eigenvalues: Symmetric Matrices

Given a symmetric matrix $A \in \mathbb{S}^n$

- The eigenvalues of A are real, i.e., $\lambda_i \in \mathbb{R}$
- Eigenvectors of A are orthonormal, i.e., matrix U of eigenvectors is *orthogonal*.

The diagonalized form of $A \in \mathbb{S}^n$ is also called **eigen decomposition**

$$A = U\Lambda U^{-1} = U\Lambda U^T, \Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_n)$$

Given any vector $x \in \mathbb{R}^n$

$$x^T A x = x^T U\Lambda U^T x = (U^T x)^T \Lambda U^T x = y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2$$

Inverse of a Symmetric Matrix

Given a symmetric matrix $A \in \mathbb{S}^n$, we know that the matrix of its eigenvectors is orthogonal and full rank, i.e., $U^{-1} = U^T$. If *all* the eigenvalues are **non-zero**, i.e., A is full rank, using the eigen decomposition we can write

$$A^{-1} = (U \Lambda U^T)^{-1} = (U^T)^{-1} \Lambda^{-1} U^{-1} = U \Lambda^{-1} U^T$$

where $\Lambda^{-1} = \text{Diag}(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n})$

Eigenvalues and Definiteness

The symmetric square matrix $A \in \mathbb{S}^n$ is

- positive definite if $\lambda_i > 0$
- positive semidefinite if $\lambda_i \geq 0$
- negative definite if $\lambda_i < 0$
- negative semidefinite if $\lambda_i \leq 0$
- indefinite if it has both positive and negative eigenvalues

Additional Material and Examples

Quadratic Forms II

From the equation above it can be concluded that a_{ij} and a_{ji} contribute to the quadratic form in the same way.
Since $x^T Ax$ is a scalar

$$x^T Ax = (x^T Ax)^T = x^T A^T x = x^T (\frac{1}{2}A + \frac{1}{2}A^T)x$$

where $B = \frac{1}{2}A + \frac{1}{2}A^T$ is a symmetric matrix.

- If $D = \mathrm{Diag}(d_1, \dots, d_n)$

$$x^T Dx = \sum_{i=1}^{n} d_i x_i^2$$

Singular Values I

Remember Eigenvalues of a symmetric square matrix $A \in \mathbb{S}^n$ are real.

For a matrix $A \in \mathbb{R}^{m \times n}$ the product $A^T A \in \mathbb{R}^{n \times n}$ is a square symmetric matrix

- The eigenvalues of $A^T A$ are real
- The eigenvalues of $A^T A$ are non-negative
  proof. If $x \in \mathbb{R}^n$ is an eigenvector of $A^T A$ and λ is its corresponding eigenvalue, we know that $A^T A x = \lambda x$. Therefore,

  $$x^T A^T A x = (Ax)^T (Ax) = \|Ax\|_2^2 = x^T \lambda x = \lambda \|x\|_2^2$$

  Since $\|Ax\|_2^2 \geq 0$ and $\|x\|_2^2 \geq 0$, we conclude that $\lambda \geq 0$

Singular Values II

Given the matrix $A \in \mathbb{R}^{m \times n}$, denote $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$(may be repeated) as the eigenvalues of $A^T A$. The singular values of matrix A are the square root of the eignevalues of $A^T A$

$$\sigma_i = \sqrt{\lambda_i}, \quad 1 \leq i \leq n$$

- The rank of matrix $A \in \mathbb{R}^{m \times n}$ is equal to the number of its non-zero singular values

Singular Value Decomposition(SVD)

Given a matrix $A \in \mathbb{R}^{m \times n}$ and its non-zero singular values $\sigma_1, \ldots, \sigma_r$, it can be decomposed(not unique) as

$$A = U \Sigma V^T$$

- $U \in \mathbb{R}^{m \times m}$ is orthogonal $U^T U = I$
- $V \in \mathbb{R}^{n \times n}$ is orthogonal $V^T V = I$
- $\Sigma \in \mathbb{R}^{m \times n}$

$$\Sigma_{ii} = \sigma_i, \quad 1 \leq i \leq r$$
$$\Sigma_{ij} = 0, \quad \text{otherwise}$$

Thin SVD

Given the matrix $A \in \mathbb{R}^{m \times n}$ with rank equal r, the thin SVD can be represented as

$$A = U \Sigma V^T$$

- $U \in \mathbb{R}^{m \times r}$ has the only r columns corresponding to non-negative singular values
- $V \in \mathbb{R}^{r \times n}$ has the only r columns corresponding to non-negative singular values
- $\Sigma \in \mathbb{R}^{r \times r}$ is a diagonal matrix with

$$\Sigma_{ii} = \sigma_i, \quad 1 \leq i \leq r$$
$$\Sigma_{ij} = 0, \quad \text{otherwise}$$

SVD: Finding Orthogonal Matrices U and V

Eigendecomposition of the symmetric matrix $A^T A$ can be formulated as

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = (V\Sigma^T U^T)(U\Sigma V^T) = V(\Sigma^T \Sigma)V^T$$

- The columns v_i of matrix V are eigenvectors of $A^T A$

Eigendecomposition of the symmetric matrix AA^T can be formulated as

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = (U\Sigma V^T)(V\Sigma^T U^T) = U(\Sigma\Sigma^T)U^T$$

- The columns u_i of matrix U are eigenvectors of AA^T

SVD and Eigendecomposition

If matrix $A \in \mathbb{S}_+^n$ is symmetric and positive semidefinite, the matrices U and V in the singular value decomposition are the same

$$A = U\Sigma V^T = U\Sigma U^T$$

In fact, the singular values of A are equal to its eigenvalues $\lambda_i = \sigma_i$

Example of SVD I

Given $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$, we want to find its SVD. First we compute $A^T A$

$$A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Solving

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda = 0$$

eigenvalues of $A^T A$ are

$$\lambda_1 = 5,\ \lambda_2 = 0$$

Example of SVD II

Therefore, the singular values of A are

$$\sigma_1 = \sqrt{5},\ \sigma_2 = 0$$

Solving

$$(A - \lambda_1)v_1 = \begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} v_1 = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} v_1 = 0$$

results in the first normalized eignevector $v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Example of SVD III

$A^T A$ is symmetric and its eigenvectors are orthonormal

$$v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Therefore, matrix V can be represented as

$$V = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

Example of SVD III

Computing matrix $AA^T = 1$ results in a eignenvector $u_1 = \begin{bmatrix}1\end{bmatrix}$. Therefore, the singular value decomposition of A can be represented as

$$\begin{bmatrix}1 & 2\end{bmatrix} = \begin{bmatrix}1\end{bmatrix} \begin{bmatrix}\sqrt{5} & 0\end{bmatrix} \begin{bmatrix}\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix}^T$$

Diagonalizable Matrix

Given a square matrix $A \in \mathbb{R}^{n \times n}$, its eigenvectors x_i, and its eigenvalues λ_i, two matrices $X \in \mathbb{R}^{n \times n}$ and $\Lambda \in \mathbb{R}^{n \times n}$ can be defined as

$$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_n \\ | & | & & | \end{bmatrix}, \Lambda = \text{Diag}(\lambda_1, \dots, \lambda_n)$$

We can write

$$AX = X\Lambda$$

If X is invertible(i.e, full rank), matrix A is **diagonalizable**

$$A = X\Lambda X^{-1}$$

Orthogonal Matrices Are Full Rank I

If $U \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, it is full rank, i.e., all the columns are linearly independent

$$U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \end{bmatrix}$$

proof. If the columns are not linearly independent then

$$\exists\, \alpha_1, \dots, \alpha_n, \alpha_i \neq 0\,, \alpha_i u_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j u_j$$

Orthogonal Matrices Are Full Rank II

Multiplying by u_i^T we have

$$\alpha_i u_i^T u_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j u_i^T u_j$$

Since u_i and u_j are orthogonal

$$\alpha_i \|u_i\|^2 = 0$$

which is only possible if $\alpha_i = 0$ and it is a contradiction

Positive (or Negative) Definite Matrices Are Full Rank

A positive or a negative definite matrix is always *full rank*.

proof. Suppose ith column is a linear combination of other columns

$$\exists\, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in \mathbb{R},\ a_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} x_j a_j$$

Set $x_i = -1$

$$\sum_{i=1}^{n} a_i x_i = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Ax = 0$$

Therefore, $x^T A x = 0$ for a non-zero vector x, which is a contradiction.

Eigenvalues and Definiteness: Proof

For the square matrix $A \in \mathbb{S}^n$, the matrix of eigenvectors $U \in \mathbb{R}^{n \times n}$ is full rank and invertible. Therefore, its columns span $\mathbb{R}^n$ and any vector $y \in \mathbb{R}^n$ can be represented in terms of $U^T x$

$$x^T A x = y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2$$

Since $\forall\, 1 \leq i \leq n$, $y_i^2 \geq 0$, the symmetric matrix A is

- positive definite if $\lambda_i > 0$
- positive semidefinite if $\lambda_i \geq 0$
- negative definite if $\lambda_i < 0$
- negative semidefinite if $\lambda_i \leq 0$
- indefinite if it has both positive and negative eigenvalues

Rank-Nullity Theorem

Given matrix $A \in \mathbb{R}^{m \times n}$

$$\text{rank}(A) + \text{nullity}(A) = n$$

Eigenvalues: Application to Optimization Problems

Given a symmetric matrix $A \in \mathbb{S}^n$

- The solution to the minimization problem

$$\min_{x \in \mathbb{R}^n} x^T A x, \quad \text{subject to } \|x\|_2 = 1$$

  is the eigenvector corresponding to the minimum eigenvalue

- The solution to the maximization problem

$$\max_{x \in \mathbb{R}^n} x^T A x, \quad \text{subject to } \|x\|_2 = 1$$

  is the eigenvector corresponding to the maximum eigenvalue

Example of Finding Eigenvalues I

Given $A = \begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$, we want to find its eigenvalues and eigenvectors

$$\begin{vmatrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{vmatrix} = -30+5\lambda-6\lambda+\lambda^2+18 = \lambda^2-\lambda-12 = (\lambda+3)(\lambda-4)$$

Eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 4$.

Example of Finding Eigenvalues II

Solving

$$(A - \lambda_1)x_1 = \begin{bmatrix} -5+3 & 2 \\ -9 & 9 \end{bmatrix} x_1 = \begin{bmatrix} -2 & 2 \\ -9 & 9 \end{bmatrix} x_1 = 0$$

results in the first eignevector $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solving

$$(A - \lambda_2)x_2 = \begin{bmatrix} -5+3 & 2 \\ -9 & 9 \end{bmatrix} x_2 = \begin{bmatrix} -9 & 2 \\ -9 & 2 \end{bmatrix} x_2 = 0$$

results in the second eignevector $x_2 = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$

Singular Values: Application to Optimization Problems

Given a matrix $A \in \mathbb{R}^{m \times n}$

- The solution to the minimization problem

$$\min_{x \in \mathbb{R}^n} \|Ax\|, \quad \text{subject to } \|x\|_2 = 1$$

  is the eigenvector corresponding to the minimum eigenvalue of $A^T A$

- The solution to the maximization problem

$$\max_{x \in \mathbb{R}^n} \|Ax\|, \quad \text{subject to } \|x\|_2 = 1$$

  is the eigenvector corresponding to the maximum eigenvalue of $A^T A$

Pseudo-inverse

The pseudo-inverse of a matrix $A = U\Sigma V^T$ is denoted as

$$A^\dagger = V\Sigma^{-1}U^T$$

where $\Sigma^{-1} \in \mathbb{R}^{n\times m}$ is a diagonal matrix

$$\Sigma_{ii}^{-1} = 1/\sigma_i, \quad 1 \le i \le r$$
$$\Sigma_{ij}^{-1} = 0, \quad \text{otherwise}$$

- If $m \ge n$ and A is full rank, i.e., linearly independent columns, $A^\dagger = (A^TA)^{-1}A^T$, which is also a left inverse $A^\dagger A = I$
- If $m \le n$ and A is full rank,i.e., linearly independent rows, $A^\dagger = A^T(AA^T)^{-1}$, which is also a right inverse $AA^\dagger = I$

Finding Pseudo-Inverse of the Above SVD Example

The pseudo-inverse $A^\dagger$ can be represented as

$$A^\dagger = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Since A has linearly independent rows, the pseudo-inverse is also a right inverse

$$AA^\dagger = \frac{1}{5} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$$A^\dagger A = \frac{1}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Example of SVD I

Given $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$, we want to find its SVD. First we compute $A^T A$

$$A^T A = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

Eigenvalues of $A^T A$ are

$$\lambda_1 = 360,\ \lambda_2 = 90,\ \lambda_3 = 0$$

Note Matrix $A^T A$ can have rank at most 2, therefore, it was expected that $\lambda_3 = 0$.

Example of SVD II

Therefore, the singular values of A are

$$\sigma_1 = \sqrt{360} = 6\sqrt{10},\ \sigma_2 = \sqrt{90} = 3\sqrt{10},\ \lambda_3 = 0$$

The matrix $\Sigma \in \mathbb{R}^{2\times 3}$ is represented as

$$\Sigma = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

Finding eigenvectors of $A^T A$, matrix V can be represented as

$$V = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -2/3 & 1/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

Example of SVD III

Matrix AA^T can be computed as

$$AA^T = \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}$$

Finding the eigenvalues and eigenvectors of AA^T, matrix U can be represented as

$$U = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}$$

Example of SVD IV

The singular value decomposition of A can be represented as

$$\begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} =$$

$$\begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -2/3 & 1/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}^T$$

Matrix Calculus I

If $x \in \mathbb{R}^n$ and $y = f(x) \in \mathbb{R}^m$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

- Given $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $y = Ax \in \mathbb{R}^m$

$$\frac{\partial y}{\partial x} = A$$

Matrix Calculus II

Given vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$

- $\dfrac{\partial y^T A x}{\partial x} = y^T A$
- $\dfrac{\partial y^T A x}{\partial y} = x^T A^T$

Given a square matrix $A \in \mathbb{R}^{n \times n}$

- $\dfrac{\partial x^T A x}{\partial x} = x^T (A + A^T)$
- If A is symmetric, $\dfrac{\partial x^T A x}{\partial x} = 2x^T A$