Problem 1: Maximum Likelihood Estimation

$\lambda = 0$

Assume a device has a lifetime modeled by an exponential distribution, i.e., $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ (and 0 for $x < 0$). We have tested 5 devices and their lifetimes were $2, 3, 1, 3$, and 4 years. Find the maximum likelihood estimation of λ. What happens when the test results are $0.5, 1, 2, 4, 0.5$?

λ: parameter $\quad X = \{2,3,1,3,4\}$ observation

$$\lambda^{ML} = \arg\max_{\lambda} \Pr(X \mid \lambda)$$

$$= \arg\max_{\lambda} \Pr(\{2,3,1,3,4\} \mid \lambda)$$

i.i.d

$$= \arg\max_{\lambda} \Pr(2 \mid \lambda) \Pr(3 \mid \lambda) \Pr(1 \mid \lambda) \Pr(3 \mid \lambda) \Pr(4 \mid \lambda)$$

$$= \arg\max_{\lambda} \lambda e^{-2\lambda} \times \lambda e^{-3\lambda} \times \lambda e^{-\lambda} \times \lambda e^{-3\lambda} \times \lambda e^{-4\lambda}$$

$$= \arg\max_{\lambda} \lambda^5 e^{-13\lambda}$$

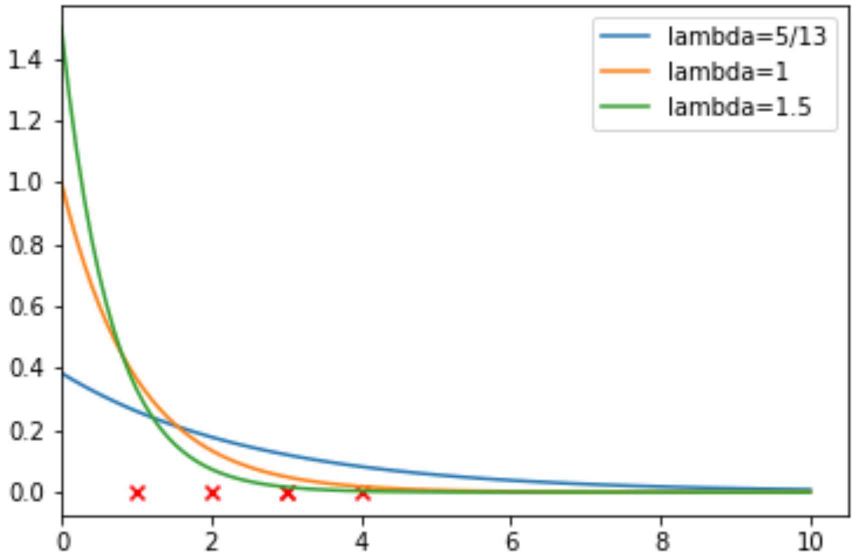
$$\frac{\partial \lambda^5 e^{-13\lambda}}{\partial \lambda} = 0 = 5\lambda^4 e^{-13\lambda} - 13\lambda^5 e^{-13\lambda} = 0$$

λ^* $\quad \lambda \neq 0$

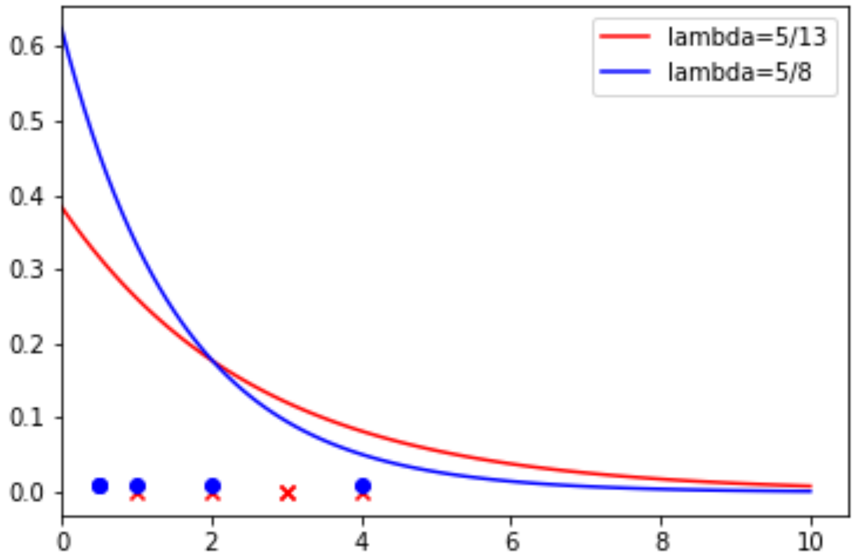
$$\Rightarrow 5 e^{-13\lambda} - 13\lambda e^{-13\lambda} = 0$$

$$\Rightarrow 5 = 13\lambda \Rightarrow \lambda = \frac{5}{13}$$

part2 $\rightarrow \frac{5}{8}$



(a) Exponential distribution for different values of λ and the sampling points



(b) Two sets of samples and the maximum likelihood estimation of λ

Figure 1: Problem 1

Problem 2: Maximum A Posteriori Estimation

For the problem above, consider an exponential prior distribution over λ with parameter 1.5, i.e., $f(\lambda) = 1.5e^{-1.5\lambda}$. Find the MAP estimation for λ.

Exercise. Find the MAP estimation for λ when we have a zero mean Gaussian prior with variance 1, i.e., $f(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}}$

$$\lambda^{MAP} = \underset{\lambda}{\operatorname{argmax}} \Pr(\lambda \mid X) = \underset{\lambda}{\operatorname{argmax}} \Pr(X \mid \lambda) \Pr(\lambda)$$

$X = \{1, 2, 3, 3, 4\}$ Prior dist: $\Pr(\lambda) = 1.5\, e^{-1.5\lambda}$

$$\underset{\lambda}{\operatorname{argmax}} \Pr(\{1,2,3,3,4\} \mid \lambda) \times 1.5\, e^{-1.5\lambda}$$

$$= \underset{\lambda}{\operatorname{argmax}} \Pr(1 \mid \lambda) \Pr(2 \mid \lambda) \Pr(3 \mid \lambda) \Pr(3 \mid \lambda) \Pr(4 \mid \lambda)\, 1.5\, e^{-1.5\lambda}$$

$$= \underset{\lambda}{\operatorname{argmax}} \left(\lambda^5 e^{-13\lambda}\right) \left(1.5 e^{-1.5\lambda}\right)$$

$$= \underset{\lambda}{\operatorname{argmax}}\ 1.5\, \lambda^5 e^{-14.5\lambda}$$

$$\rightarrow \frac{\partial \bigcirc}{\partial \lambda} = 0 = 5\lambda^4 e^{-14.5\lambda} - 14.5\, \lambda^5 e^{-14.5\lambda} = 0$$

$$\Rightarrow \lambda = \frac{5}{14.5}$$

$$\underset{\lambda}{\operatorname{argmax}} \left(\lambda^5 e^{-13\lambda}\right) \cancel{\Pr(\lambda)} \rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}}$$

$$\hookrightarrow \lambda^5 e^{-13\lambda - \frac{\lambda^2}{2}}$$

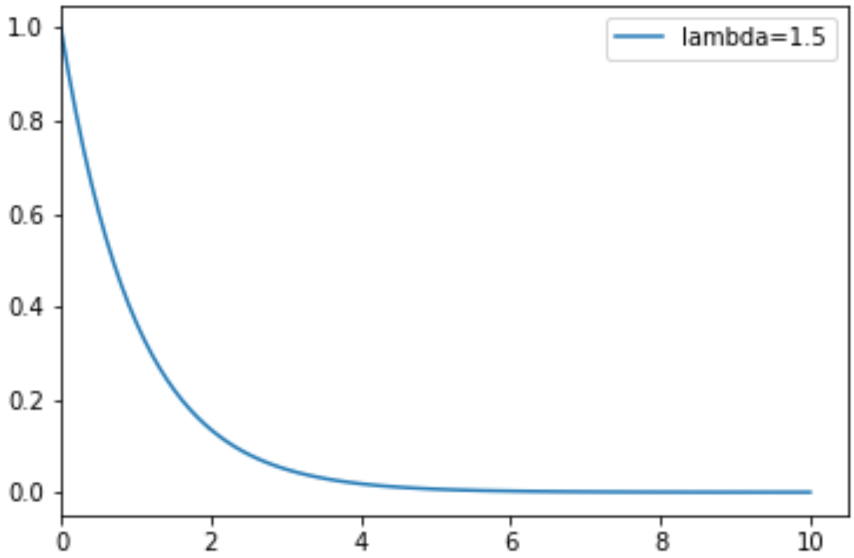


Figure 2: Prior exponential distribution with parameter 1.5