ASTRON 2E03 (Winter 2024) Assignment 2 Due: Thursday, March 7th

Be sure to include your full name and ID number in the top right corner of your submission. Failure to do so will result in an automatic deduction of 5%.

Students are to submit hard copies of their assignments. Assignments are due at the beginning of class on the due date listed at the top of this page. Students who are unable to submit their own assignment in class should make arrangements to submit their assignment early, have a class-mate submit their assignment in class, or consult with the course instructor to make alternative arrangements. Late assignments will incur a late penalty of 5% per day (including weekends). Please write legibly and show all your work to earn full points.

Students with approved accommodations who require an extension should make their extension request at least one week prior to the due date.

Q1) Lagrange point L4 [17 points]

In class we set up the three-body problem according to Figure 1 below. We then showed that the effective potential on the test mass m is

$$\Phi = -G\left(\frac{M_1}{s_1} + \frac{M_2}{s_2}\right) - \frac{1}{2}\omega^2 r^2, \tag{1}$$

where M_1 and M_2 are the masses of a star and planet, respectively, such that $M_1 \gg M_2$. The star and planet orbit their common centre-of-mass (COM). s_1, s_2, and r are the position vector lengths summarized in Figure 1 below, and ω is the shared angular velocity of the star and planet.

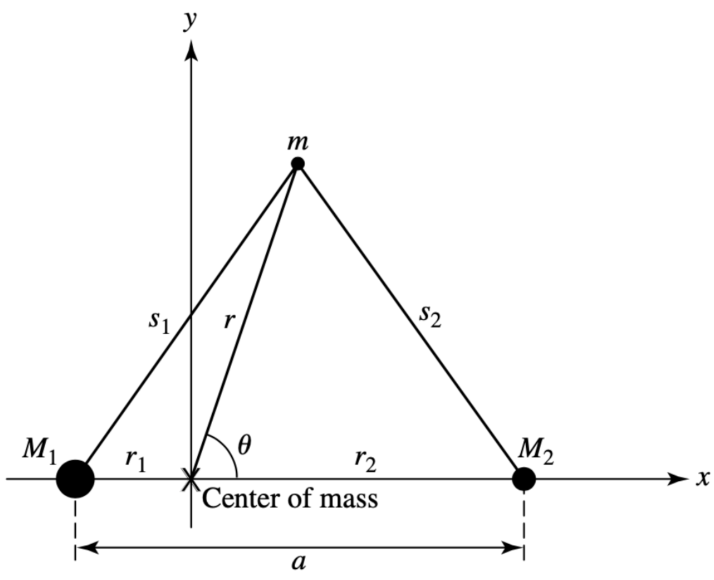

Figure 1:

In class, we argued that the Lagrange point L4 is located at the vertex of an equilateral triangle with M_1 and M_2 at the other two vertices. In this question, you will show that the effective potential at L4 is a constant. The question is broken up into the steps below to help guide you.

a) [1 point] At L4, what is relation between s_1, s_2, r_1, and r_2.

b) [2 points] Using the COM equation, rewrite your relation from part a) in terms of M_1 and M_2.

c) [5 points] The sum of the first two terms in Eq. 1 is the total gravitational potential acting on m. Use your results from part b) to simplify the gravitational potential to a single term. Is this term a constant?

d) [7 points] Turning now to the centrifugal term: show that the centrifugal potential is a constant in the limit that the mass of one of the massive bodies is much larger than the other (i.e. $M_1 \gg M_2$).

e) [2 points] Write out the total effective potential at L4. Comment on the implication that the potential being constant at L4 has on the motion of a test mass at L4.

Q2) Greenhouse effect: thermal inversion [19 points]
Consider a 1D atmosphere model with two atmospheric layers over the planet's surface. Each atmospheric layer is completely opaque to longwave radiation emitted by the surface and by the other atmospheric layers.

a) [5 points] Sketch the incoming and outgoing sources of radiation for each of the atmospheric layers and surface in this model. Note that your diagram should be similar in style to the GH toy model depicted in the Lecture 5 slides.

b) [6 points] Imagine a scenario in which the deep atmospheric layer is unable to radiate away all of its energy. To be precise, let's assume that it can only radiate away half of the available energy in each direction. The remainder goes into heating the atmospheric layer. Conversely, the uppermost atmospheric layer can radiate all of its energy in each direction. The surface can also radiate all of its available energy upward. Write down the energy budget for the surface and both atmospheric layers by equating each layer's irradiance "in" to its irradiance "out".

c) [4 points] Derive the temperature for each atmospheric layer in terms of the surface temperature T_{surf}.

d) [4 points] Sketch the corresponding diagram of temperature versus height. Note that temperature should be represented on the x-axis of your plot.

Q3) Pressure profile of an incompressible atmosphere [9 points]
In class we derived the pressure and density profiles of a planetary atmosphere that is composed of an ideal gas. However, there exists a range of pressures deep inside a giant planet's atmosphere wherein hydrogen gas undergoes a phase change and becomes liquid metallic hydrogen. This fluid is not well-described as an ideal gas and instead can be approximated as an incompressible fluid.

a) [1 point] What is the equation of state of an incompressible fluid (i.e. what is the scaling between an incompressible fluid's density ρ and its pressure P)?

b) [6 points] Derive the corresponding pressure profile $P(r)$ of an incompressible fluid, where r is the depth in the atmosphere.

c) [2 point] Sketch the pressure profile as a function of r. Note that pressure should be represented on the x-axis of your plot.