ENG 3PX3 - Engineering Economics

McMaster University

Nonlinear Optimization

Solver Review

→Recall that we can use Solver to find an optimal value (maximum, minimum, or target value) for a formula in the objective cell

	A	B	C	D	E	F	G
1	**Oil Refinery**						
2							
3	Country	Variable	#bbl	Cost/bbl	Gas	Jet	Lubricant
4	Canada	x1	2608.696	$ 50.00	0.3	0.4	0.2
5	USA	x2	3043.478	$ 37.50	0.4	0.15	0.35
6	**Objective Function**						
7	Cost	$244,565.22	Forumla: =C4*D4+C5*D5				
8							
9	**Constraints**				Formula		
10	Gas Demand	2000	>=	2000	C4*E4+C5*E5		
11	Jet Demand	1500	>=	1500	C4*F4+C5*F5		
12	Lubricant Demand	1586.956522	>=	1000	C4*G4+C5*G5		
13	Canada Limit	2608.695652	<=	9000	C4		
14	USA Limit	3043.478261	<=	6000	D4		
15	Non-negative x1	2608.695652	>=	0	C4		
16	Non-negative x2	3043.478261	>=	0	D4		

Solver Review

- →Recall that we can use Solver to find an optimal value (maximum, minimum, or target value) for a formula in the objective cell
- →Solver adjusts the decision variable cells to compute the formulas in the objective and constraint cells

	A	B	C	D	E	F	G
1	**Oil Refinery**						
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4	Canada	x1	2608.696	$ 50.00	0.3	0.4	0.2
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11	Jet Demand	1500	>=	1500	C4*F4+C5*F5		
12	Lubricant Demand	1586.956522	>=	1000	C4*G4+C5*G5		
13	Canada Limit	2608.695652	<=	9000	C4		
14	USA Limit	3043.478261	<=	6000	D4		
15	Non-negative x1	2608.695652	>=	0	C4		
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Solver Review

- →Recall that we can use Solver to find an optimal value (maximum, minimum, or target value) for a formula in the objective cell
- →Solver adjusts the decision variable cells to compute the formulas in the objective and constraint cells
- →**It will adjust the values in the decision variable cells to satisfy the constraints and produce the optimal solution**

	A	B	C	D	E	F	G
1	**Oil Refinery**						
2							
3	Country	Variable	#bbl	Cost/bbl	Gas	Jet	Lubricant
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12	Lubricant Demand	1586.956522	>=	1000	C4*G4+C5*G5		
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Nonlinear Functions

→Consider the following NVF with 3 independent decision variables, ***D***, ***L***, and ***V:***

$$NV = \$532(1 - cV') - \frac{\$3L'D'^2}{V'} - \$32\frac{L' - 0.25\sqrt{L'}}{D'^4V'^2}$$

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$$NV = \$532(1 - cV') - \frac{\$3L'D'^2}{V'} - \$32\frac{L' - 0.25\sqrt{L'}}{D'^4V'^2}$$

where each variable is unitless and defined relative to a starting or default value, i.e.,

$$D' \equiv \frac{D}{2\text{ mm}}, L' \equiv \frac{L}{30\text{ cm}}, \text{ and } V' \equiv \frac{V}{5\text{ V}}$$

and these relative variables are allowed the following ranges:

$$D' \in (0.05,8), L' \in (0.1,10), V' \in (0.2,4),$$

(and $c = 0.02$ is a parameter).

Nonlinear Functions

→**Consider the following NVF with 3 independent decision variables, *D*, *L*, and *V*:**

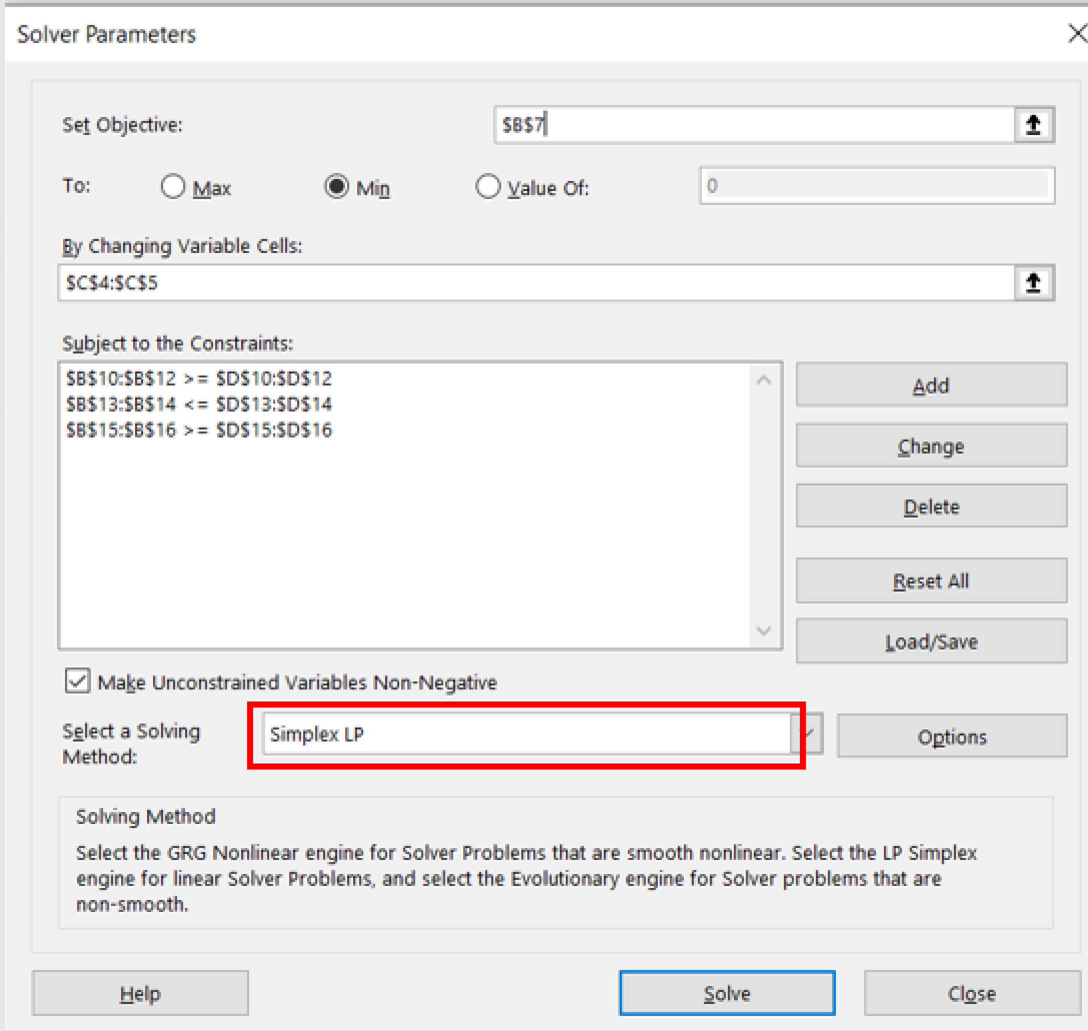
$$NV = \$532(1 - cV') - \frac{\$3L'D'^2}{V'} - \$32\frac{L' - 0.25\sqrt{L'}}{D'^4V'^2}$$

→**This objective function is *nonlinear* in these variables (it isn't just a linear combination of them,** $c_1D + c_2L + c_3V$**)**

- This means that Simplex LP won't work
  - ...but there are other solvers!

Solving Methods on Excel

1. **LP Simplex**
   - Used for linear models,
     - e.g., $NV = 500x_1 + 15{,}000x_2$
2. **Generalized Reduced Gradient (GRG) Nonlinear**
   - Used for continuous, smooth nonlinear models
     - e.g., $NV = 500x_1x_2 + 15{,}000x_2^2$
3. **Evolutionary**
   - Used for discontinuous, non-smooth models
     - e.g., Use of IF, COUNT, CEILING, etc.
   - or continuous ones with multiple local extrema
     - e.g., $NV = \frac{3}{(x-2)^2+1} + \frac{4}{(x-8)^2+1}$



Recall: Simplex LP (Solver Option #1)

→A model in which the objective cell and all constraints are linear functions of the decision variables

→Linear models will always be convex and are usually easier to solve than nonlinear models

→Since all the constraints are linear, the global optimal solution will lie at an "extreme point" where two or more constraints intersect

→In Simplex LP, it is always possible to determine whether the model has:

1. No feasible solution,
2. An unbounded objective, or
3. A globally optimal solution.

GRG Nonlinear (Solver Option #2)

→**Generalized Reduced Gradient (GRG) Nonlinear is an algorithm used for models in which at least one of the constraints (or the objective) is a smooth nonlinear function of the decision variables**

→**Nonlinear constraints can make the feasible region have concave boundaries (which means simplex won't work even if the objective is linear)**

→**GRG approach:**

- Compute gradient at trial solution and move in direction of negative (when minimizing) or positive (when maximizing) gradient
- (do complex things to optimize how much of a step you take based on things like how quickly the gradient is changing)

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GRG Nonlinear (Solver Option #2)

→**GRG methods can normally only find a *local optimal solution***

- Based on the starting point of the decision variables, it can get stuck at a local optimum
  - The multistart option can increase the chance of finding a global optimal solution

→**Solver will iterate until either:**

- The maximum number of iterations (ran out of tries) is met
- The step size is smaller than the defined tolerance (got as close as we asked)

GRG Nonlinear Example

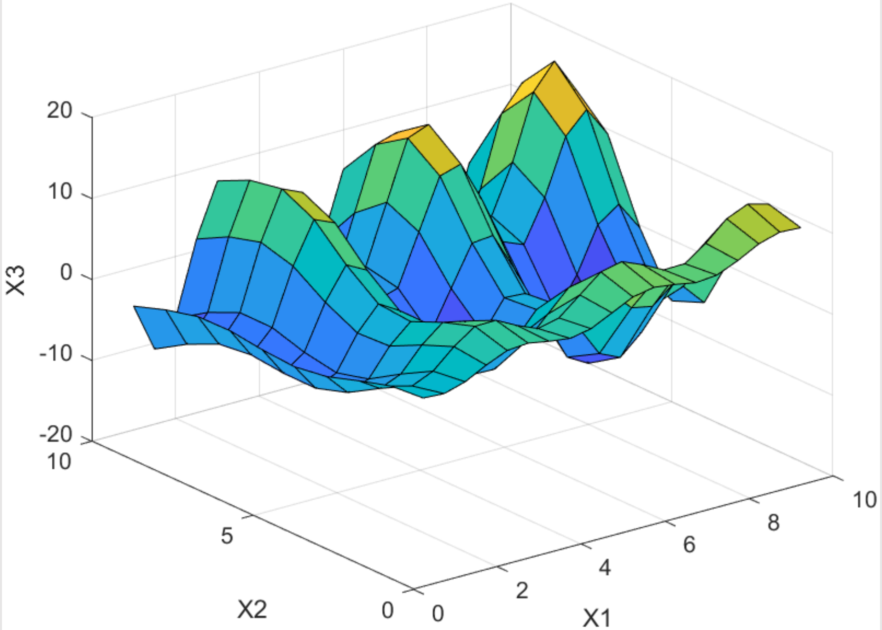
→Consider the following nonlinear model:

$$\max_{x_1,x_2} \phi = x_2 \cos(2x_1) + x_1 \sin(x_2)$$

s.t.

$$x_1, x_2 \leq 10$$

$$x_1, x_2 \geq 1$$



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GRG Nonlinear Example

→We can express the model in Excel as:

	A	B	C	D	E	F
1	Decision Variables					
2		X1=	1			
3		X2=	1			
4						
5	Objective Function					
6		Y=	0.425324	"=C3*COS(2*C2)+C2*SIN(C3)"		
7						
8	Constraints					
9	X1	1	<=	10		
10	X1	1	>=	1		
11	X2	1	<=	10		
12	X2	1	>=	1		
13						

GRG Nonlinear Example

→We will run the model with the following three starting values:

- $x_1, x_2 = (1,1)$
- $x_1, x_2 = (5,5)$
- $x_1, x_2 = (9,9)$

→Notice how we get different results with each trial run:

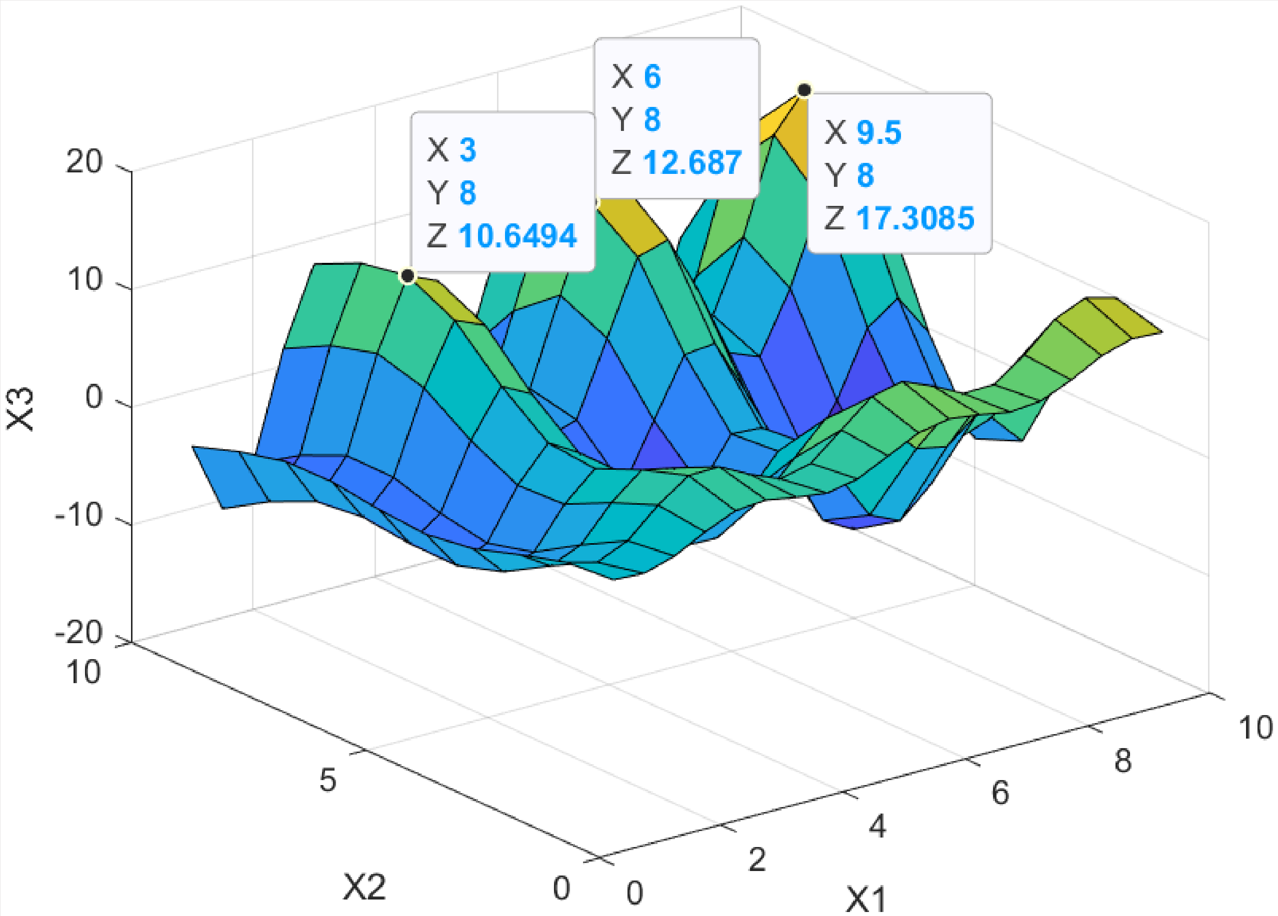
	A	B	C	D	E	F
1	Decision Variables					
2		X1=	1			
3		X2=	1.141593			
4						
5	Objective Function					
6		Y=	0.434227	"=C3*COS(2*C2)+C2*SIN(C3)"		
7						
8	Constraints					
9	X1	1	<=	10		
10	X1	1	>=	1		
11	X2	1.141593	<=	10		
12	X2	1.141593	>=	1		

	A	B	C	D	E	F
1	Decision Variables					
2		X1=	6.314007			
3		X2=	8.01272			
4						
5	Objective Function					
6		Y=	14.23213	"=C3*COS(2*C2)+C2*SIN(C3)"		
7						
8	Constraints					
9	X1	6.314007	<=	10		
10	X1	6.314007	>=	1		
11	X2	8.01272	<=	10		
12	X2	8.01272	>=	1		

	A	B	C	D	E	F
1	Decision Variables					
2		X1=	9.456031			
3		X2=	7.959724			
4						
5	Objective Function					
6		Y=	17.34739	"=C3*COS(2*C2)+C2*SIN(C3)"		
7						
8	Constraints					
9	X1	9.456031	<=	10		
10	X1	9.456031	>=	1		
11	X2	7.959724	<=	10		
12	X2	7.959724	>=	1		

GRG Nonlinear Example

→Our initial guesses $(1,1), (5,5), (9,9)$ each resulted in a different local maxima!



Evolutionary (Solver Option #3)

→**Discontinuous and non-smooth nonlinear models should use the evolutionary solver**

- e.g., if the function is non-smooth or discontinuous (so may not always have a gradient)
- e.g., if you have multiple local optima points that could confuse the GRG.

Evolutionary (Solver Option #3)

→The Evolutionary Solver:

- uses random sampling to generate a population of trial solutions
- refines where to generate the next generation of samples based on 'fitness' of the trial solutions
- Keeps going until the current solution stops getting better
- Relies on randomness, so each run may result in different answers or speeds
- More *robust* than GRG (i.e., less easily fooled and less sensitive to initial conditions depending on the objective), but typically *slower* than GRG (takes more iterations) when the function is well behaved.

Example:

→**Consider the following NVF with 3 independent decision variables, *D*, *L*, and *V*:**

$$NV = \$532(1 - cV') - \frac{\$3L'D'^2}{V'} - \$32\frac{L' - 0.25\sqrt{L'}}{D'^4V'^2}$$

where each variable is unitless and defined relative to a starting or default value, i.e.,

$$D' \equiv \frac{D}{2\text{ mm}}, L' \equiv \frac{L}{30\text{ cm}}, \text{ and } V' \equiv \frac{V}{5\text{ V}}$$

and these relative variables are allowed the following ranges:

$D' \in (0.05,8), L' \in (0.1,10), V' \in (0.2,4)$, (and $c = 0.02$ is a parameter).

Which solver is best for this?

Example:

→**Consider the following NVF with 3 independent decision variables, *D*, *L*, and *V*:**

$$NV = \$532(1 - cV') - \frac{\$3L'D'^2}{V'} - \$32\frac{L' - 0.25\sqrt{L'}}{D'^4V'^2}$$

→**This objective function is *nonlinear* in these variables (it isn't just a linear combination of them,** $c_1D + c_2L + c_3V$**)**

- This means that Simplex LP won't work

→**But it is well behaved in the allowed range of variables (no discontinuities)**

- → GRG nonlinear is *likely* best, but if there are multiple local extrema it could be tricked and we'll need to use GRG with multistart or evolutionary.

Nonlinear Functions

→**Consider the following NVF in terms with 3 independent decision variables to choose from, *D*, *L*, and *V*:** $NV = \$532(1 - cV') - \frac{\$3L'D'^2}{V'} - \$32\frac{L' - 0.25\sqrt{L'}}{D'^4V'^2}$

- Setup and solved with GRG →

H2 | =532*(1-E2*B4)-3*B3*B2^2/B4-32*(B3-0.25*SQRT(B3))/B2^4/B4^2

	A	B	C	D	E	F	G	H	I
1	Variable			Parameter			Objective		
2	D'	1.520797		c	0.02		NVF	525.2756	
3	L'	0.1							
4	V'	0.361139							
5									
6									
7									
8	Constraints								
9	D'	0.05	<=	1.520797	<=	8			
10	L'	0.1	<=	0.1	<=	10			
11	V'	0.2	<=	0.361139	<=	4			

Nonlinear Functions

→**Consider the following NVF in terms with 3 independent decision variables to choose from, *D*, *L*, and *V*:** $NV = \$532(1 - cV') - \frac{\$3L'D'^2}{V'} - \$32\frac{L' - 0.25\sqrt{L'}}{D'^4 V'^2}$

- Setup and solved with GRG →

How can we find out how sensitive the optimum (inputs and NV) is to parameters (like c) if we can't use Simplex LP and get a sensitivity report?

3	L'	0.1				
4	V'	0.361139				
5						
6						
7						
8	Constraints					
9	D'	0.05	<=	1.520797	<=	8
10	L'	0.1	<=	0.1	<=	10
11	V'	0.2	<=	0.361139	<=	4

Nonlinear Solver Sensitivity Analysis

→Redo the optimization at a different parameter value

H2 =532*(1-E2*B4)-3*B3*B2^2/B4-32*(B3-0.25*SQRT(B3))/

	A	B	C	D	E	F	G	H
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→

H2 =532*(1-E2*B4)-3*B3*B2^2/B4-32*(B3-0.25*SQRT(B3))/

	A	B	C	D	E	F	G	H
1	Variable			Parameter			Objective	
2	D'	1.531181		c	0.022		NVF	524.8992
3	L'	0.1						
4	V'	0.346685						
5								
6								
7								
8	Constraints							
9	D'	0.05	<=	1.531181	<=	8		
10	L'	0.1	<=	0.1	<=	10		
11	V'	0.2	<=	0.346685	<=	4		

Increase parameter, *c*, by 10% and redo the optimization

Nonlinear Solver Sensitivity Analysis

→Redo the optimization at a different parameter value

H2 =532*(1-E2*B4)-3*B3*B2^2/B4-32*(B3-0.25*SQRT(B3))/

	A	B	C	D	E	F	G	H
1	Variable			Parameter			Objective	
2	D'	1.520797		c	0.02		NVF	525.2756
3	L'	0.1						
4	V'	0.361139						
5								
6								
7								
8	Constraints							
9	D'	0.05	<=	1.520797	<=	8		
10	L'	0.1	<=	0.1	<=	10		
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→

H2 =532*(1-E2*B4)-3*B3*B2^2/B4-32*(B3-0.25*SQRT(B3))/

	A	B	C	D	E	F	G	H
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2	D'	1.531181		c	0.022		NVF	524.8992
3	L'	0.1						
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8	Constraints							
9	D'	0.05	<=	1.531181	<=	8		
10	L'	0.1	<=	0.1	<=	10		
11	V'	0.2	<=	0.346685	<=	4		

New optimum values for decision variables

Nonlinear Solver Sensitivity Analysis

→Redo the optimization at a different parameter value

H2 =532*(1-E2*B4)-3*B3*B2^2/B4-32*(B3-0.25*SQRT(B3))/

	A	B	C	D	E	F	G	H
1	Variable			Parameter			Objective	
2	D'	1.520797		c	0.02		NVF	525.2756
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→

H2 =532*(1-E2*B4)-3*B3*B2^2/B4-32*(B3-0.25*SQRT(B3))/

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9	D'	0.05	<=	1.531181	<=	8		
10	L'	0.1	<=	0.1	<=	10		
11	V'	0.2	<=	0.346685	<=	4		

New optimal value for objective function

Nonlinear Solver Sensitivity Analysis

→Redo the optimization at a different parameter value

H2 — fx =532*(1-E2*B4)-3*B3*B2^2/B4-32*(B3-0.25*SQRT(B3))/

	A	B	C	D	E	F	G	H
1	Variable			Parameter			Objective	
2	D'	1.520797		c	0.02		NVF	525.2756
3	L'	0.1						
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→

H2 — fx =532*(1-E2*B4)-3*B3*B2^2/B4-32*(B3-0.25*SQRT(B3))/

	A	B	C	D	E	F	G	H
1	Variable			Parameter			Objective	
2	D'	1.531181		c	0.022		NVF	524.8992
3	L'	0.1						
4	V'	0.346685						

Determine the effect of changing the parameter on NV and decision variable:

$$\frac{\partial NV_{@opt}}{\partial c} \approx \frac{\Delta NV_{@opt}}{\Delta c} = \frac{524.8992 - 525.2756}{0.022 - 0.02} = -188.2$$

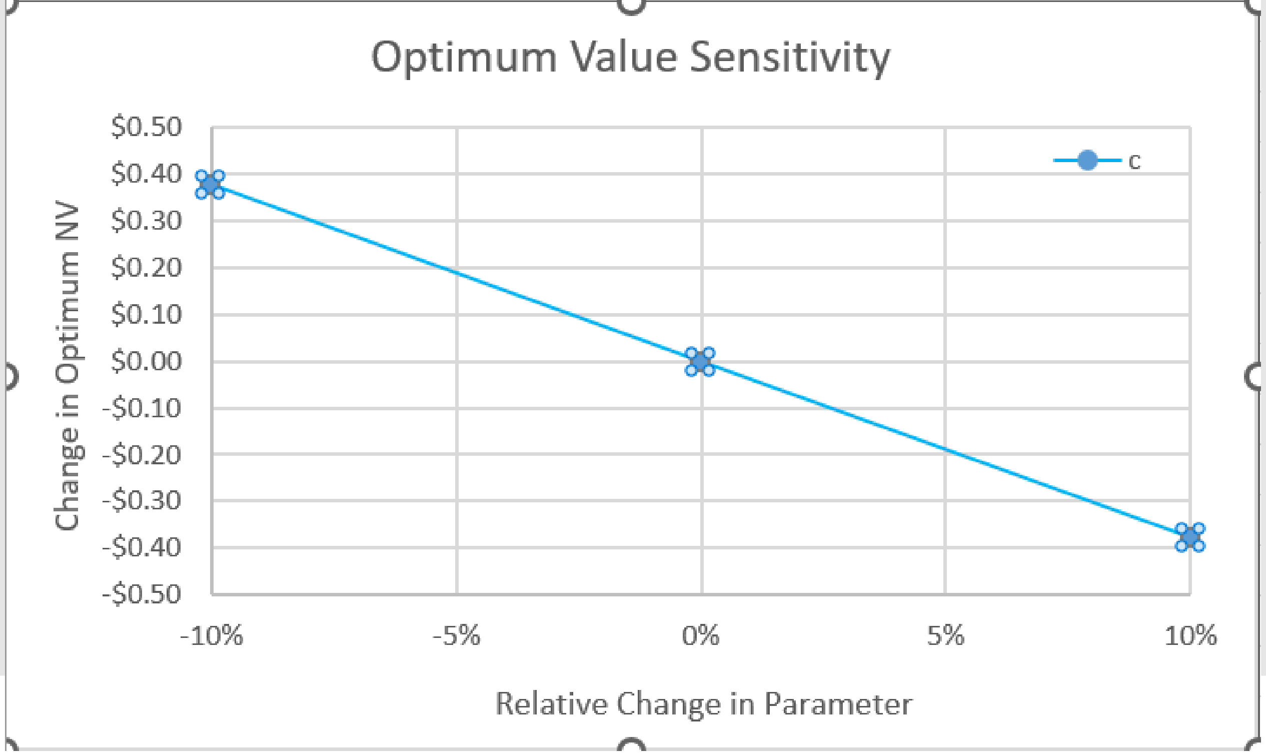
similarly,

$$\frac{\partial D'}{\partial c} \approx \frac{\Delta D'}{\Delta c} = \frac{1.531181 - 1.520791}{0.022 - 0.02} = 5.195$$

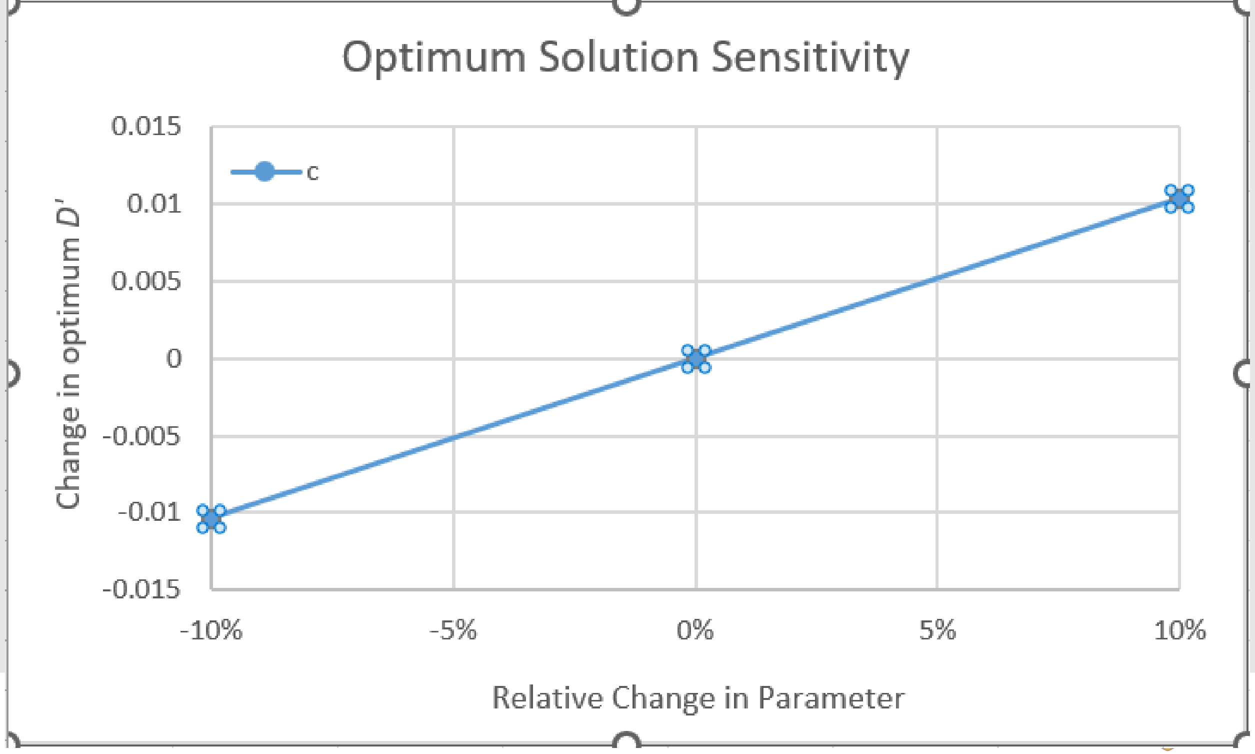
Nonlinear Solver Sensitivity Analysis

→Spider plot has optimum's change for % increase & decrease of parameter (not slope directly)

Sensitivity of Optimum				Spider Plot Setup		
c	NV	D'		c	DeltaNV	DeltaD'
0.02	$525.28	1.520797		-10%	$0.38	-0.01038
0.022	$524.90	1.531181		0	$0.00	0
Slope:	-$188.19	5.192265		10%	-$0.38	0.010385
	=(B18-B19	=(C18-C19)/(A18-A19)				



Sensitivity of Optimum				Spider Plot Setup		
c	NV	D'		c	DeltaNV	DeltaD'
0.02	$525.28	1.520797		-10%	$0.38	-0.01038
0.022	$524.90	1.531181		0	$0.00	0
Slope:	-$188.19	5.192265		10%	-$0.38	0.010385
	=(B18-B19	=(C18-C19)/(A18-A19)				



Dealing with non-numeric decision variables

→Recall the NVF for nanoRIMS we worked with in Lecture 6 (to make spider & tornado plots):

$$NV = \frac{\$896}{\text{week}} - \left(C_{ingred} + C_{space} + C_{time} + C_{device}\right)$$

$$NV = \frac{\$896}{\text{week}} - \left(\frac{\$5}{100\text{ mL}} \times q_{ingred} + \frac{\$12.5}{\text{hr}} \times \frac{1}{8} \times t_{FumeHood} + \frac{\$15}{\text{hr}} \times t_{GradStudent} + C_{device} + \frac{\$10}{52} \times \frac{\rho\dot{\mathcal{V}}\left(\frac{32\dot{\mathcal{V}}}{\pi D^4}\left(\frac{\dot{\mathcal{V}}}{\pi} + 12\nu L\right) + g\Delta z\right)}{10\ \mathbf{mW}} + \$1.1875\frac{\pi D^2}{\mathbf{cm}^2}\right)$$

Is it possible to use a solver with “non-numeric” decision variables???

e.g., suppose we believe we can add a self-correction system to nanoRIMS for a cost increase of $400/yr reducing Grad Student time required by 1 hr/wk. How could we modify the function to consider this non-numeric decision variable (add self-correction system or don’t)?

Dealing with non-nun

Note that with evolutionary the lower & upper bounds need to be specified with direct references (B11:B12) and not indirect ones (G11:G12)

Warning – takes a long time!

	A	B	C	D	E	F	G	H	I
1	**Tech Analysis Parameters:**			nanoRIMS					
2	pi	3.14		Objective:					
3	V (flow rate)	5.83E-06	m3/s	NV (rel to Purchasing)	$652.71	/week	=B26-SUMPRODUCT(B17:B20,E17:E20)-B21-B22		
4	ρ	1000	kg/m3						
5	v (kinematic visc)	1.00E-06	m2/s	**Constraints**	LB		var		UB
6	g	9.81	m/s2	D	1.00E-03	<=	3.53E-03	<=	1.00E-02
7	z	0.2	m	Self-Correction	0	<=	1.00E+00	<=	1
8	L	3.00E-01	m						
9									
10	**Direct Input Decision Variables:**								
11	D	3.53E-03	m						
12	Self-correction	1	Whether to add $400/yr and reduce GS time by 1 hr /week						
13									
14	**Indirect Decision Variables and more parameters:**								
15	W	1.97E+00	W						
16					per week needed				
17	Ingredients	$5	/100 mL	=5	2	100 mL			
18	Fume Hood Time	$1.56	/hr	=12.5/8	98	hr	=14*7		
19	Device	$1,000	/yr	=600+IF(B12=1,400,0)	0.019231		=1/52		
20	Grad Student Time	$15	/hr	=15	1.5	hr	=2.5-IF(B12=1,1,0)		
21	Cost pump	$38	/wk	=B15/0.01*10/52					
22	Cost tube uncertainty	$0	/wk	=1.1875*B2*B11^2*100^2					
23									
24	Purchasing								
25	Cost	$112	/25 mL						
26	Total Cost	$896	/week