ENG 3PX3 - Engineering Economics

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Optimization – Problem Formulation

NVF, Mathematical Models, and Optimization

→**Given a set of parameters and constraints, how do we optimize all or part of our NVF?**

→**We will look briefly at model-based optimization**

- We will not dive deep into the theory, rather we will provide an overview of optimization and show how to use Excel to perform optimization

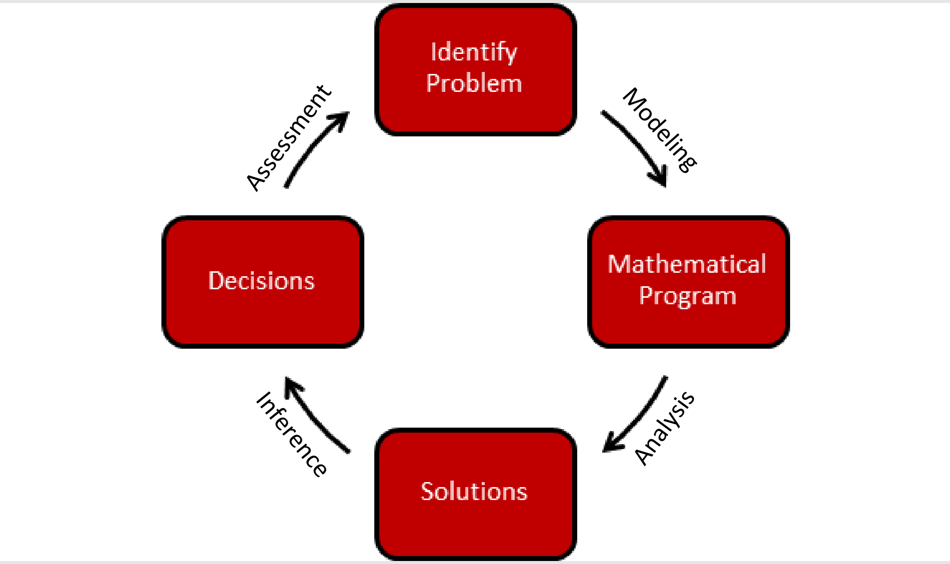
→**Given a problem, we develop a mathematical model to describe the problem.**

- Models include variables, constraints, objective function
- Example: Maximize net value considering labour, material, environment etc.

What is Model-based Optimization?

→Models are not perfect, they are a simplification of the real problem

- In model-based optimization, we draw conclusions from the *model* of the system, NOT from the problem itself
- It is critical to **develop an appropriate model for your system/problem/scenario**



→Model-based optimization is typically iterative

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Components of an Optimization Model

→ **Optimization models (mathematical programs) typically include the following components**

- **Decision Variables:** (i.e., independent variables or terms dependent on them) What you're going to change (directly or indirectly) to find the best value of the objective you can.
- **Constraints:** Set allowed values of decision variables (i.e., allowed inputs, i.e., the "domain" of the objective function).
- **Objective:** The thing you will optimize (maximize or minimize) (i.e., NV)
  - Note: technically the objective is also a decision variable, just not one that you directly decide.
- **Objective Function:** A mathematical function that determines the objective as a function of the decision variables (i.e., the NVF).

Components of an Optimization Model

→Typical set up for a generalized optimization program:

$\min\limits_{\boldsymbol{x}} \phi = f(\boldsymbol{x})$	← **Objective Function**
s.t.	← "Subject to"
$h(\boldsymbol{x}) = 0$	← Equality Constraints
$g(\boldsymbol{x}) \leq 0$	← Inequality Constraints
$\boldsymbol{x}_{lb} \leq \boldsymbol{x} \leq \boldsymbol{x}_{ub}$	← Variable Bounds

Components of an Optimization Model

→**Parameters:**

- Constants; terms you don’t get to decide when optimizing

→**Decision Variables:**

- **Independent Variables (i.e., “direct” decision variables)**
  - Things you can change directly, your inputs
  - Example: hourly wage and labour hours
- **Dependent Variables (i.e., “indirect” decision variables)**
  - Convenient combinations of independent variables that you might use as an intermediate term (or the objective itself).
  - Example: total labour cost (the product of hourly wage and labour hours)
  - Though it’s possible to write the objective in terms of only independent variables, it likely leads to a very tough to read objective function where the physical meaning of terms is hidden

Decision variables can be discrete or continuous

→**Discrete variables, $\boldsymbol{x}_D$**

- Limited to a fixed or countable set of values (all-or-nothing, either-or, must be produced/purchased in integer quantities, *etc*.)

$$\boldsymbol{x}_D \; \epsilon \; \boldsymbol{I} = \{0, 1, 2, \dots\}$$

→**Continuous variables, $\boldsymbol{x}_C$**

- May take on *any value* in a specified interval

$$\boldsymbol{x}_C \; \epsilon \; \boldsymbol{S} \subset \mathbb{R}$$ Where "⊂" means "is a subset of"

→**It is much easier to model with continuous variables than with discrete variables**

→**If a discrete variable scale is large enough that rounding to the nearest integer causes *minimal loss of model accuracy*, then we can model that variable *continuously***

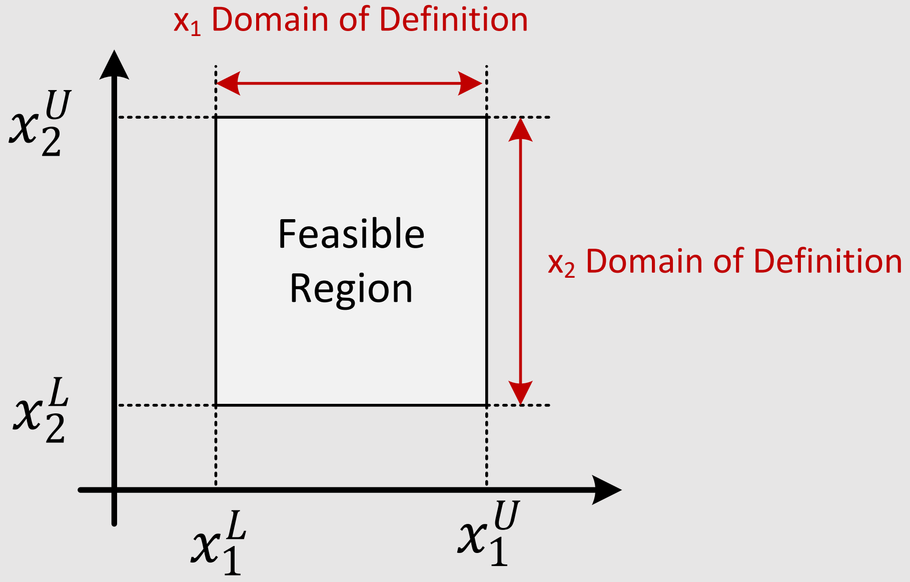
Constraints

→Decision variables are usually *bounded* (finite allowed range) because of:

- **Physical reality**: e.g., cannot purchase negative raw materials, cannot have a temperature less than 0 K
- **Model assumptions**: e.g., cannot sleep less than 6 hours per day, pipe ID is around 1/8”

→The *domain of definition* is defined by a decision variable’s upper (x^U) and lower bounds (x^L)

- The combination of domains of definition for all decision variables is the **feasible region** or **feasible set**



Graphing Model Constraints

→The feasible set (or region) $\mathcal{S}$ of an optimization model is the collection of decision variables that satisfy *all* the model constraints:

$$\mathcal{S} \triangleq \{\boldsymbol{x} : g(\boldsymbol{x}) \leq 0, h(\boldsymbol{x}) = 0, \boldsymbol{x}^L \leq \boldsymbol{x} \leq \boldsymbol{x}^U\}$$

→The set of all points satisfying $h(x) = 0$ results in a line or vector when plotting the feasible set.

→The set of all points satisfying $g(x) \leq 0$ results in a region bounded above (≤) or below (≥) by a line defining the inequality.

Graphing Model Constraints Examples

Constraint set (A)

$x_1 + x_2 \leq 2$

$3x_1 + x_2 \geq 3$

$x_1\,, x_2 \geq 0$

Constraint set (B)

$x_1 + x_2 \leq 2$

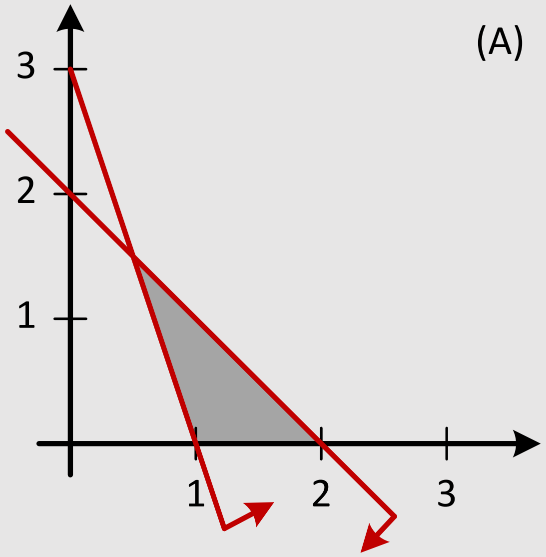
$3x_1 + x_2 = 3$

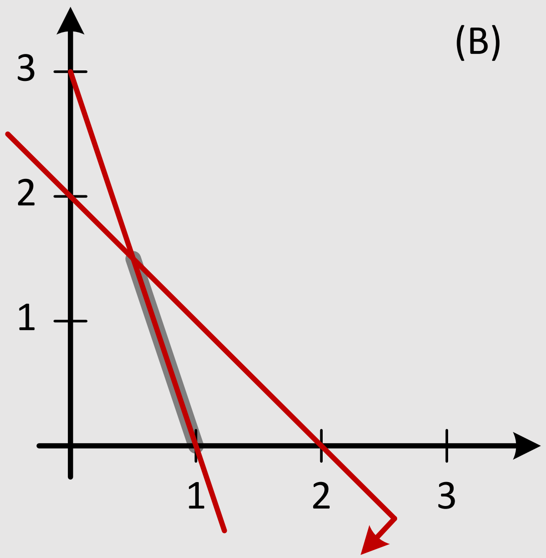
$x_1\,, x_2 \geq 0$

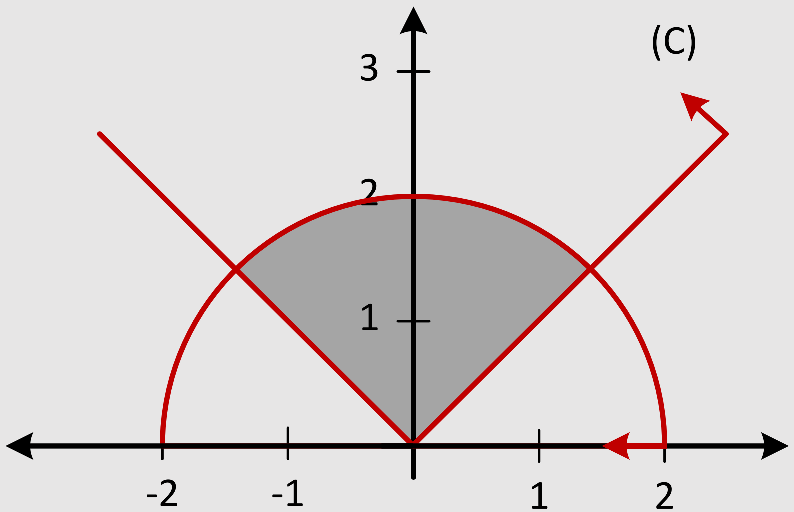
Constraint set (C)

$x_1^2 + x_2^2 \leq 4$

$|x_1| - x_2 \leq 0$







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Constraints + Objective Visualizing

Problem (A)

$$\min_{x_1,x_2} \phi = 3x_1 + x_2$$

$$s.t.$$

$$x_1 + x_2 \leq 2$$

$$x_1 , x_2 \geq 0$$

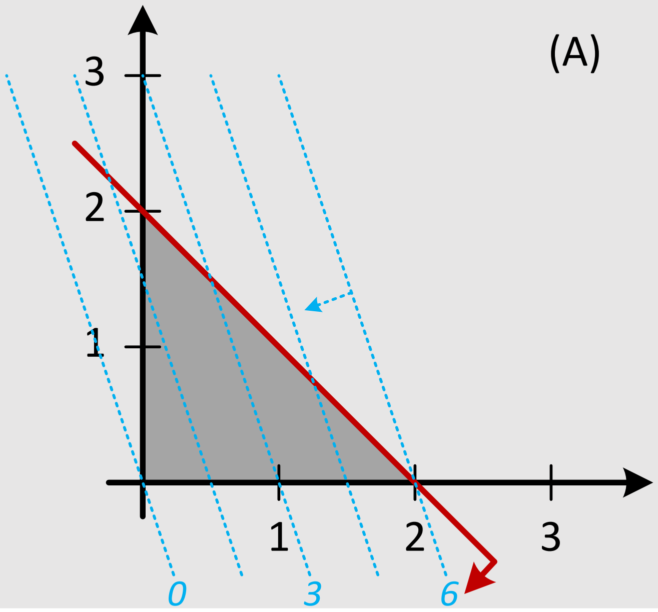
Problem (B)

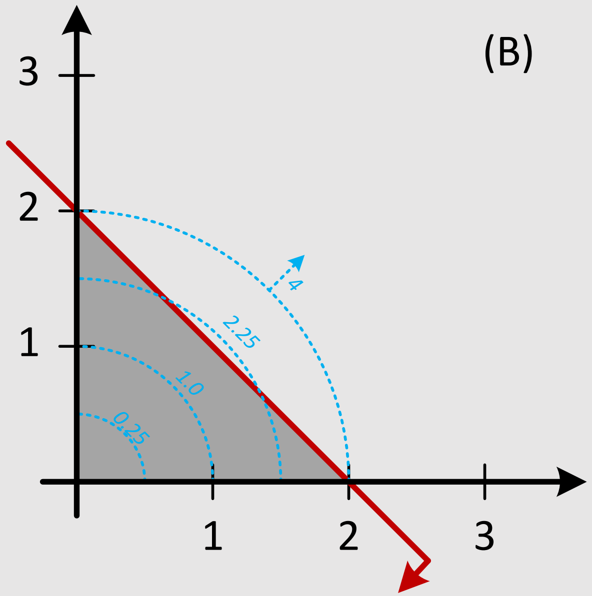
$$\max_{x_1,x_2} \phi = x_1^2 + x_2^2$$

$$s.t.$$

$$x_1 + x_2 \leq 2$$

$$x_1 , x_2 \geq 0$$





Optimization Outcomes

→**Optimal solutions are points lying on the best objective function contour that intersects with at least one boundary of the feasible region.**

→**The *optimal value* $\phi^\star$ is defined to be the value of the objective at the optimum(s): $\phi^\star \triangleq \phi(\boldsymbol{x}^\star)$.**

→**An optimization model can have only one true optimal value. It may have:**

- A **unique** optimal solution.
- Several **alternative** solutions $\boldsymbol{x}^\star$ yielding the *same* optimal $\phi^\star$.
- **No** optimal solutions (either the problem is unbounded or infeasible).

Optimization Outcomes

Considering the same optimization problems as before:

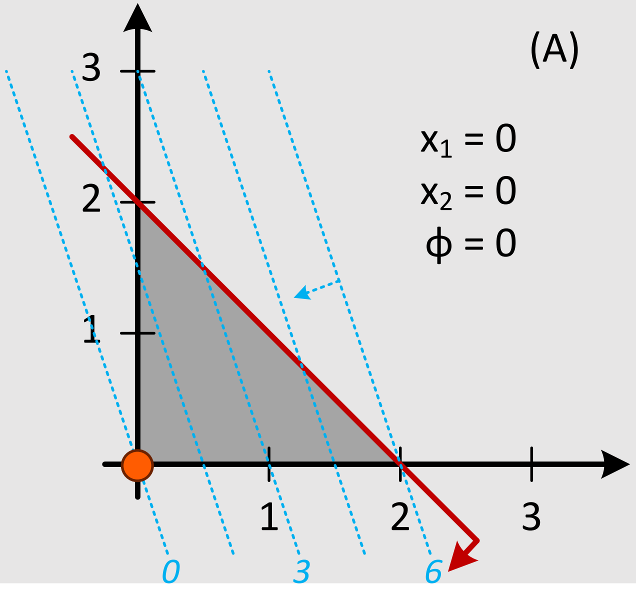
Problem (A)

$$\min_{x_1,x_2} \phi = 3x_1 + x_2$$

$$s.t.$$

$$x_1 + x_2 \leq 2$$

$$x_1\,, x_2 \geq 0$$



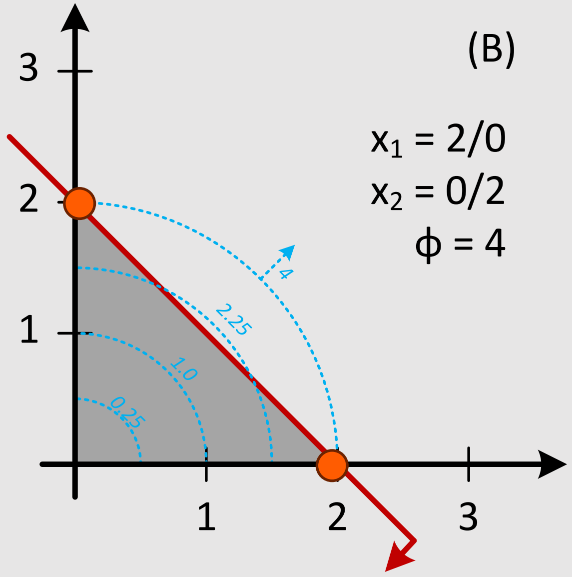
Problem (B)

$$\max_{x_1,x_2} \phi = x_1^2 + x_2^2$$

$$s.t.$$

$$x_1 + x_2 \leq 2$$

$$x_1\,, x_2 \geq 0$$



Optimization Outcomes

→Some optimization problems have degeneracy and unboundedness

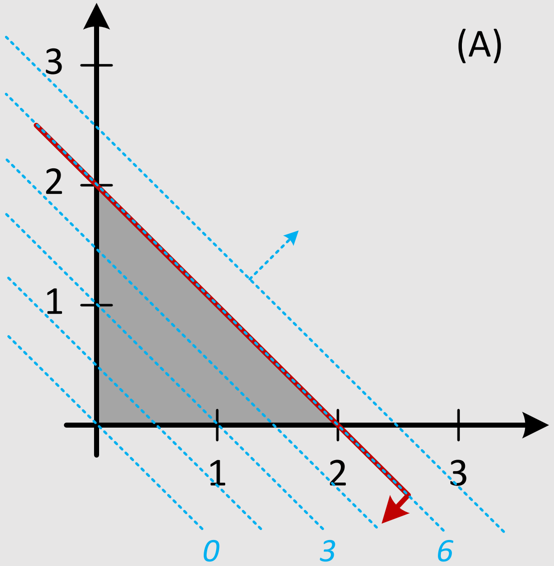
Problem (A)

$$\max_{x_1,x_2} \phi = 3x_1 + 3x_2$$

$$s.t.$$

$$x_1 + x_2 \leq 2$$

$$x_1, x_2 \geq 0$$



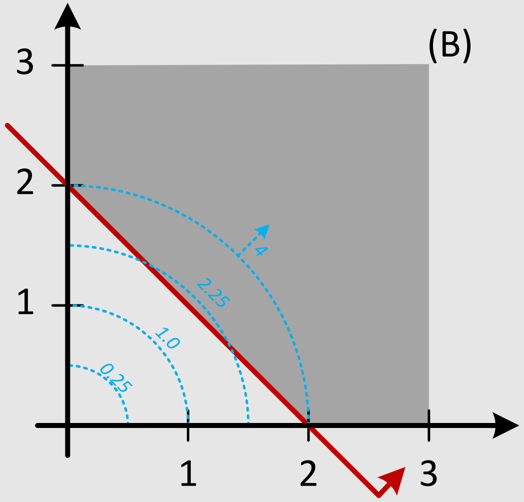
Problem (B)

$$\max_{x_1,x_2} \phi = x_1^2 + x_2^2$$

$$s.t.$$

$$x_1 + x_2 \geq 2$$

$$x_1, x_2 \geq 0$$



Comments on Constraints

→**A constraint $g(x) \leq 0$ is said to be:**

- **Active** (or **binding**) at some point $\boldsymbol{x}^\star$ if $g(\boldsymbol{x}^\star) = 0$.
- **Inactive** at some point $\boldsymbol{x}^\star$ if $g(\boldsymbol{x}^\star) < 0$.

→**Active constraints:**

- The set of constraints that are active at the optimal solution are known as the **active set**.
- Equality constraints are **ALWAYS** active at any feasible optimal point.
- No constraints (inequality or equality) may be violated at any optimal point.

Optimization Formulation Example

→**A refinery distills crude petroleum into three products: gasoline, jet fuel, and lubricants.**

→**Your plant receives crude oil shipments from Canada and USA. You want to minimize the cost.**

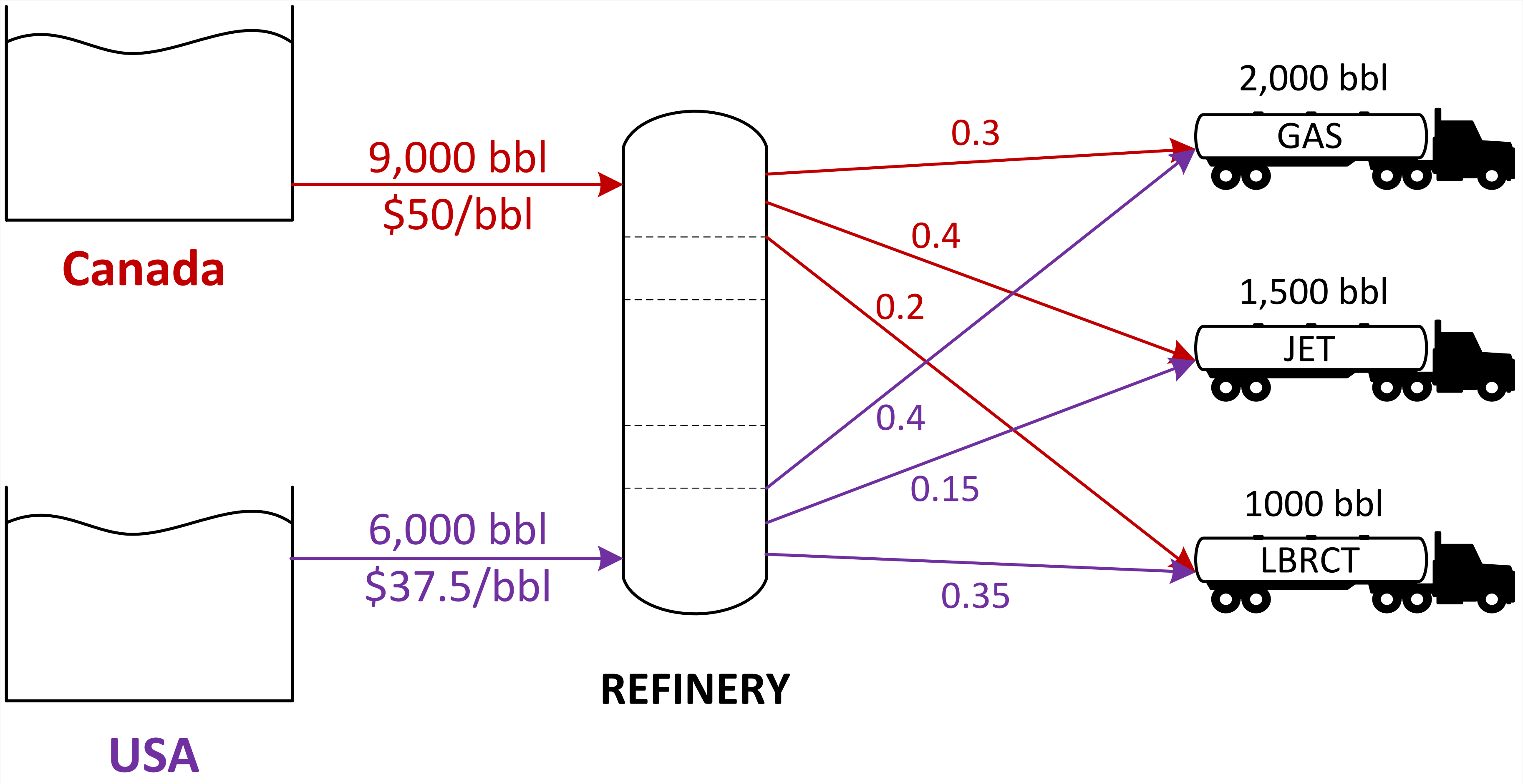
- Each barrel from Canada yields 0.3 barrels of gasoline, 0.4 barrels of jet fuel, and 0.2 barrels of lubricants.
- Each barrel from USA yields 0.4 barrels of gasoline, 0.15 barrels of jet fuel, and 0.35 barrels of lubricants.
- The remaining 0.1 (10%) from both sources is lost to the refining process.
- The Canadian oil costs your refinery $50 per barrel and is available up to 9,000 barrels per day.
- The American crude costs your refinery $37.5 but only available up to 6,000 barrels per day.

→**You have a contract with local distributors to provide per day**

- 2,000 barrels of gasoline
- 1,500 barrels of jet fuel
- 1000 barrels of lubricant

Linear Optimization Example

→Draw a diagram of the supply network



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Optimization Formulation Example

→Formulate the optimization problem:

$$
\begin{aligned}
\min_{x_1,x_2} \phi &= 50x_1 + 37.5x_2 \\
&\text{Subject to} \\
0.3x_1 + 0.4x_2 &\geq 2{,}000 \\
0.4x_1 + 0.15x_2 &\geq 1{,}500 \\
0.2x_1 + 0.35x_2 &\geq 1{,}000 \\
x_1 &\leq 9{,}000 \\
x_2 &\leq 6{,}000 \\
x_i &\geq 0 \qquad (\forall\, i)
\end{aligned}
$$

Optimization Example

→For this problem, we can start with a graphical model:

- Plot the constraints and objective function
- Visually identify the optimal value and feasible set