Fundamentals

SFWRENG 2CO3: Data Structures and Algorithms

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Collection types

A *collection type* is an *data type* used to manage a *collection of values*.

Typically an *abstract data type*: the implementation is hidden from the user.

Collection types are implemented via *data structures*.

Collection types: Bag

A *bag* B is a collection to which values can be added, but not removed:

- $\textsc{Add}(B, v)$ add value v to a bag B;
- $\textsc{Empty}(B)$ return `true` if bag B holds no values;
- $\textsc{Size}(B)$ returns the number of values in B.

In addition, one can *iterate* over the values currently in bag B.

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Remark

Note that Empty can be implemented via Size.

Not all data structures provide an efficient Size, however!

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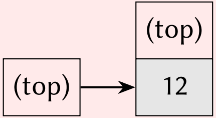

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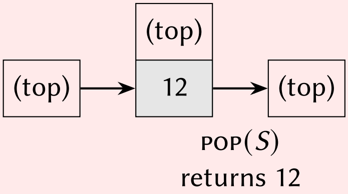

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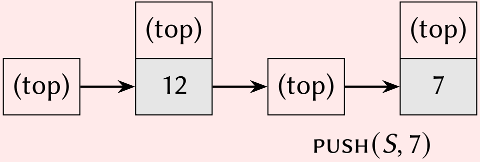

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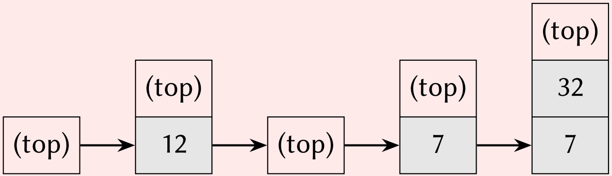

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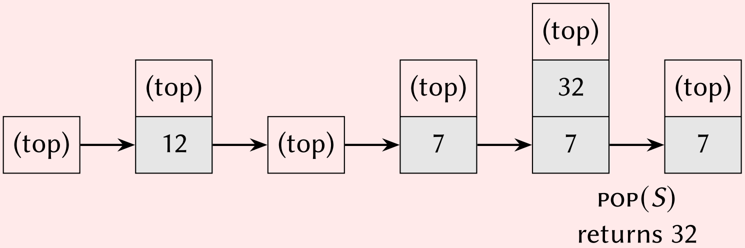

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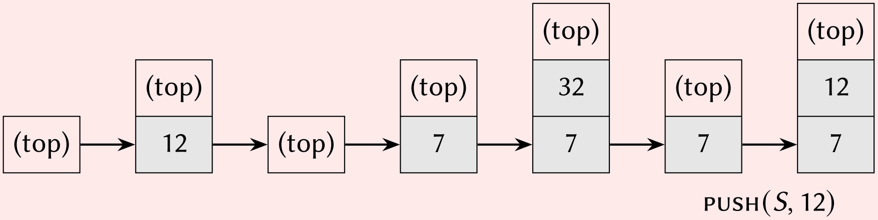

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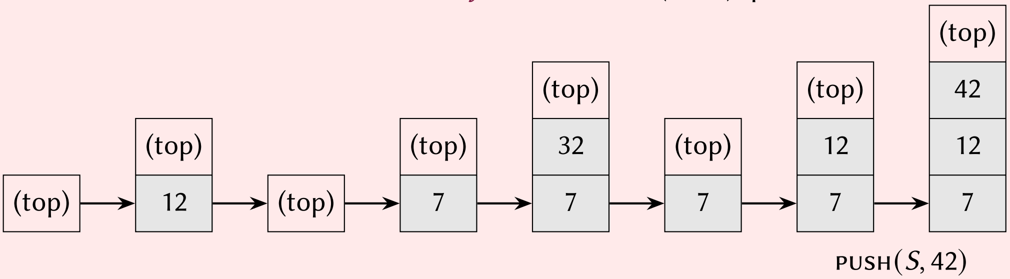

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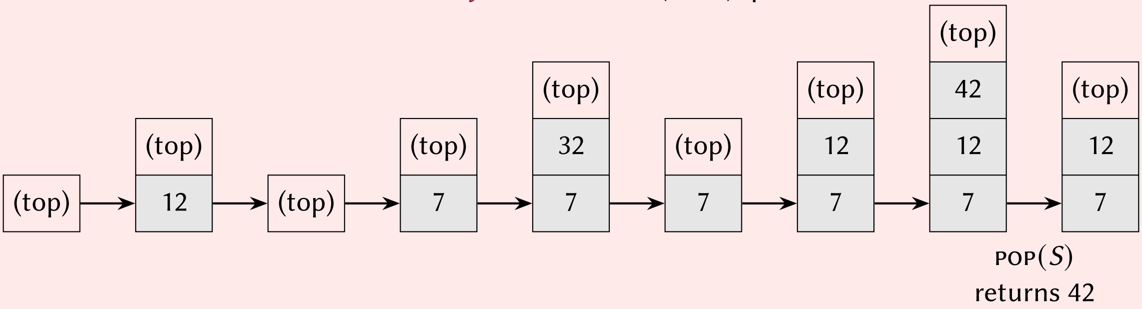

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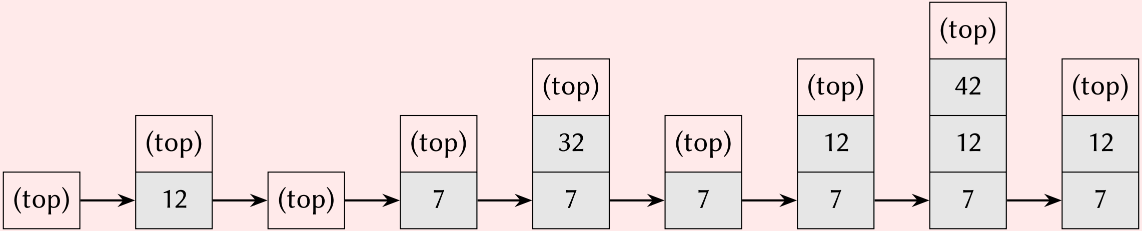

Stacks are used *everywhere*: e.g., function calls are implemented via stacks.

Using stacks to evaluate expressions in postfix notation

Postfix notation is a notation in which *operators* follow their *operands*.

Example

- “1 2 +” is equivalent to $1 + 2$.
- “1 2 3 + -” is equivalent to $1 - (2 + 3)$.
- “1 2 + 3 -” is equivalent to $(1 + 2) - 3$.
- “1 2 3 + 4 5 · - /” is equivalent to $1/((2 + 3) - (4 \cdot 5))$.

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Expressions in postfix notation are very easy to evaluate using a stack.

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Algorithm EVALUATEPN(e):

Input: e is in postfix notation.

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S := an empty stack.
for each term t ∈ e do
    if t is a value then
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5:     **else** t is an operation $\oplus$
6:         v_2 := pop(S); v_1 := pop(S).
7:         Push(S, $v_1 \oplus v_2$).
8: **return** pop(S).

(top)

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(top)
3
2
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(top)	
5	$2 + 3$
1	

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4
5
1

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20	$4\cdot 5$
5	
1	

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-15	$5-20$
1	

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$-.0\overline{6}$	$1/-15$

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Dijkstra's Two-Stack Algorithm (book) evaluates expressions in *infix* (normal) notation: this by building a postfix notation and simultaneously applying EVALUATEPN.

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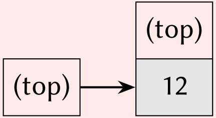

ENQUEUE(S, 12)

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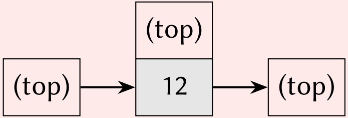

Dequeue(S)
returns 12

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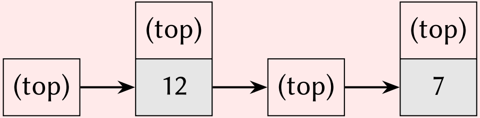

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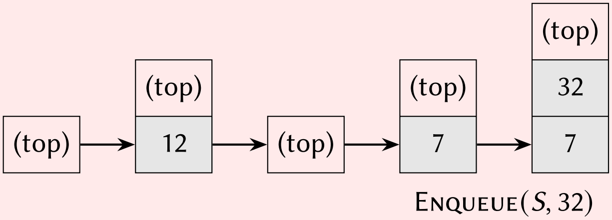

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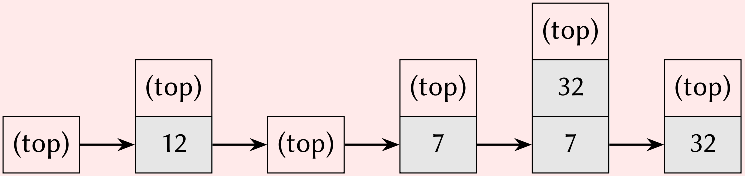

DEQUEUE(S)
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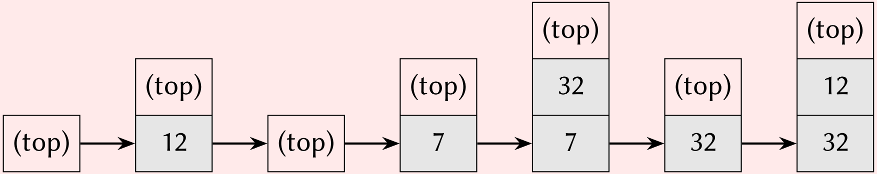

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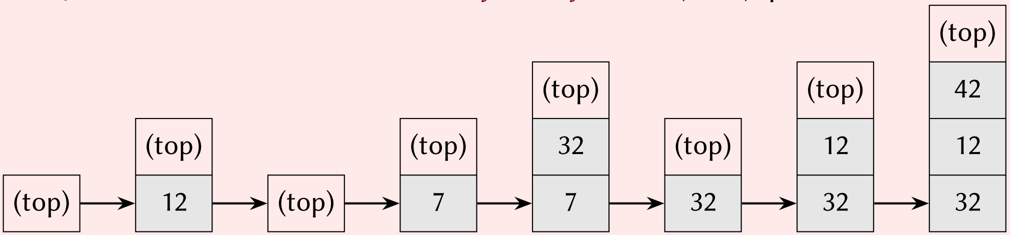

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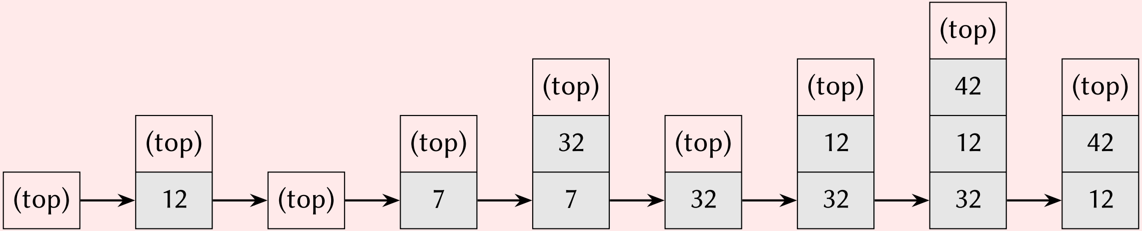

DEQUEUE(S)
returns 32

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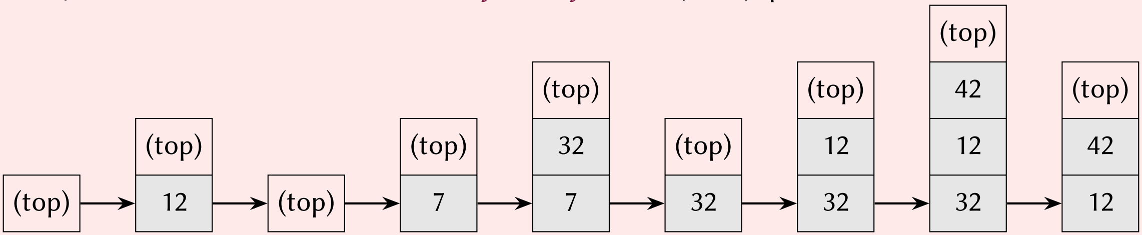

Queues are used used *everywhere*: e.g., communication buffers
(for network packages, for tasks exchanged between producer-consumer threads, ...).

Data structures: Fixed-size ring buffers

Ring Buffer: a data structure that can hold up-to-N values.

$\mathit{entries}[0 \ldots N)$	An array that can hold N values.
start	The position in *entries* of the *first* value in the buffer.
length	The current number of values in the buffer.

The values start at position *start* and wrap-around at the end of *entries*.

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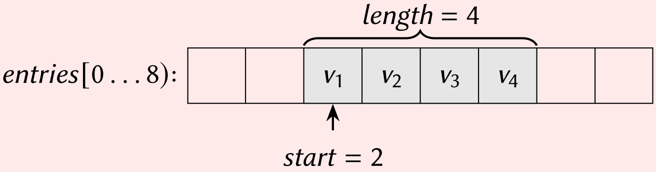

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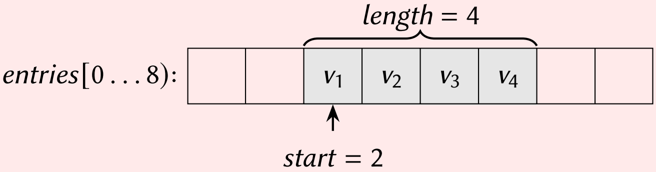

Algorithm PushBack(R, v):

Input: R is a non-full ring buffer ($R.length \neq N$).

if $R.start + R.length < N$ **then**
    $R.entries[R.start + R.length] := v$.
else Wrap-around the end of the list
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$R.length := R.length + 1$.

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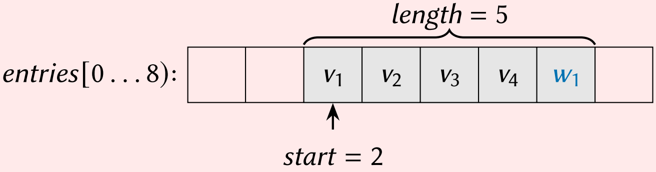

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$\textsc{PushBack}(R, w_1)$;
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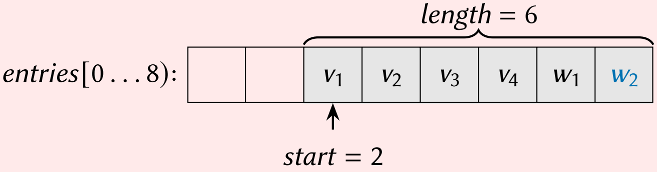

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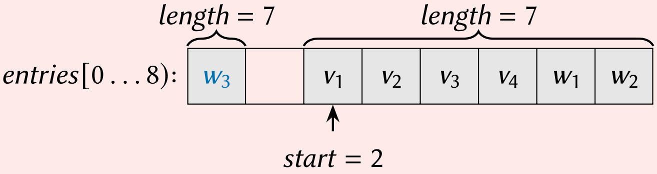

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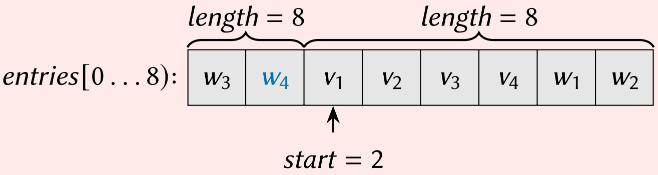

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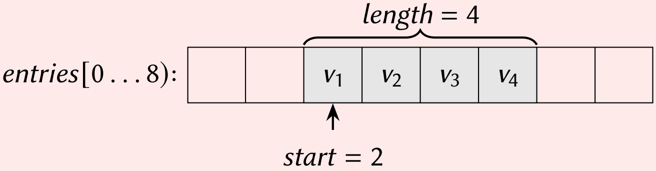

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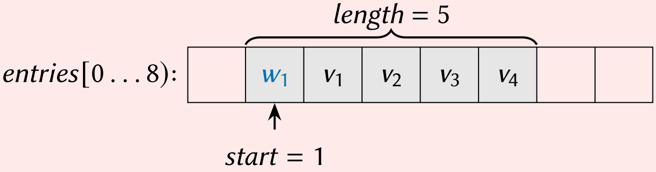

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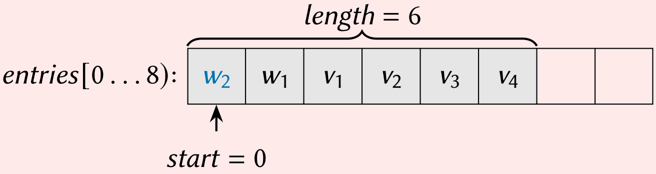

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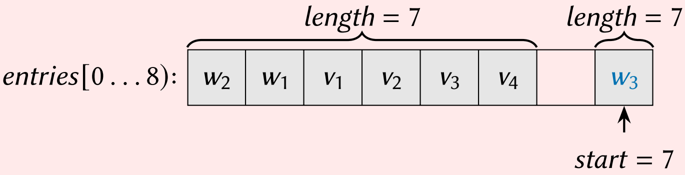

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$$\mathit{entries}[0\dots 8):\ \overbrace{\boxed{w_2}\boxed{w_1}\boxed{v_1}\boxed{v_2}\boxed{v_3}\boxed{v_4}}^{\mathit{length}\,=\,8}\overbrace{\underset{\substack{\uparrow\\ \mathit{start}\,=\,6}}{\boxed{w_4}}\boxed{w_3}}^{\mathit{length}\,=\,8}$$

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- Provides all queue operations in $\Theta(1)$.
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- Provides all stack operations in $\Theta(1)$.
- Provides all queue operations in $\Theta(1)$.
- Support *random access* efficiently.
- Can be used to implement a *double-ended queue*.
- *Drawback*: can hold at-most N values.

Singly linked lists

Linked Lists: a data structure that can hold a sequence of values, each stored in a *list node*.

first Pointer to the *first list node*.

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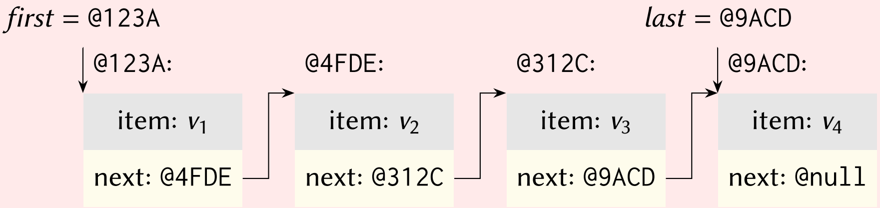

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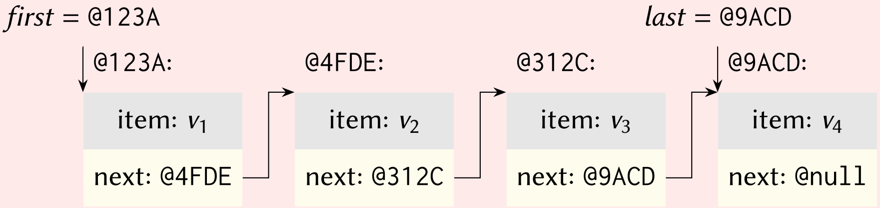

Algorithm PushFront(L, v):

Input: L is a linked list.

Create new list node n for value v.
$n.next := L.first$.
$L.first :=$ pointer to n.
if $L.last =$ `@null` **then** List L was empty
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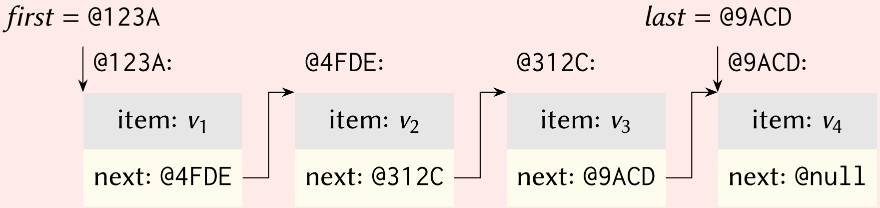

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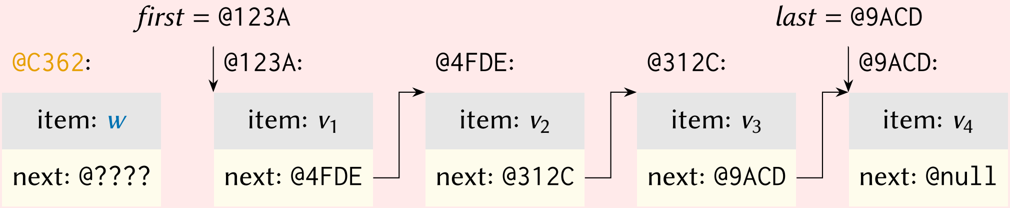

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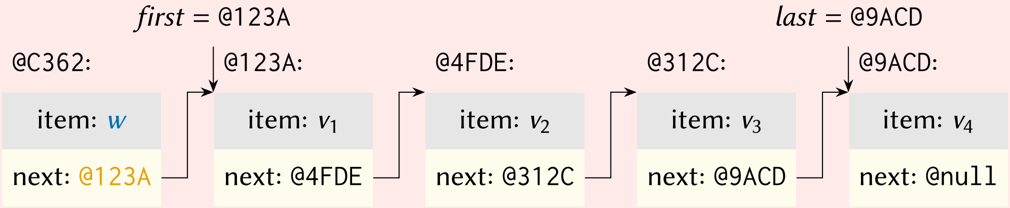

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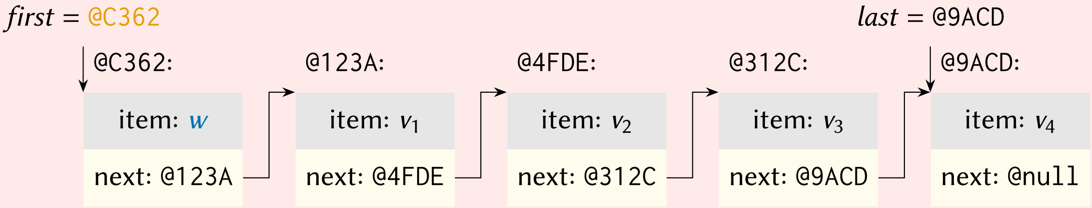

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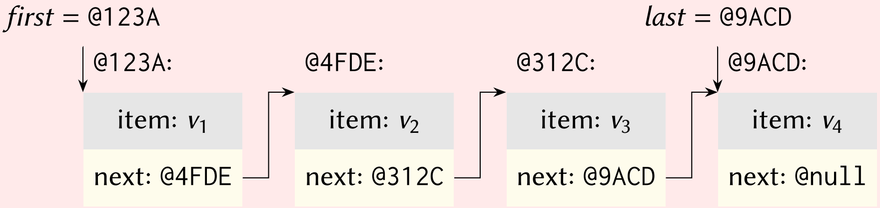

Algorithm APPENDNODE(L, m, v):

Input: L is a linked list with node m.

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$m.next :=$ pointer to n.

if $L.last = m$ **then** m was the last node in L

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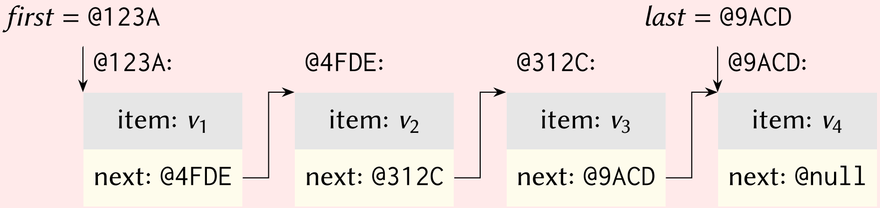

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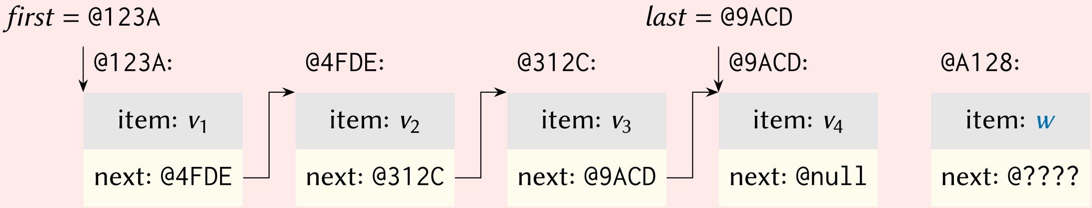

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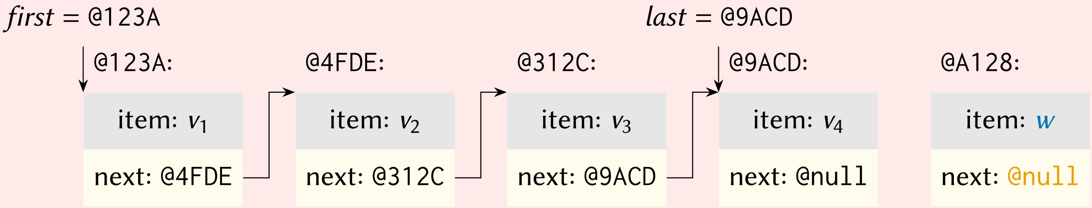

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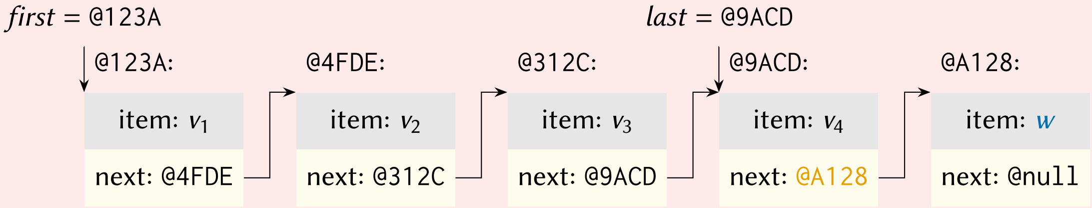

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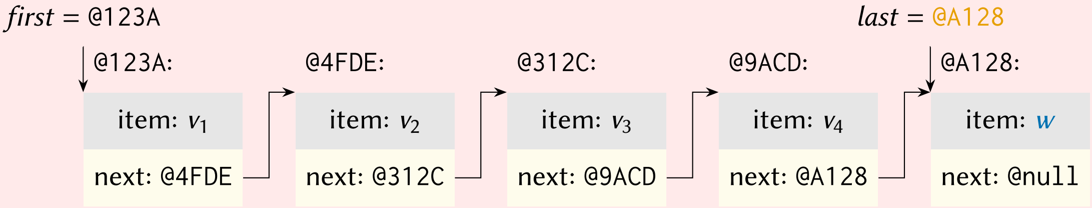

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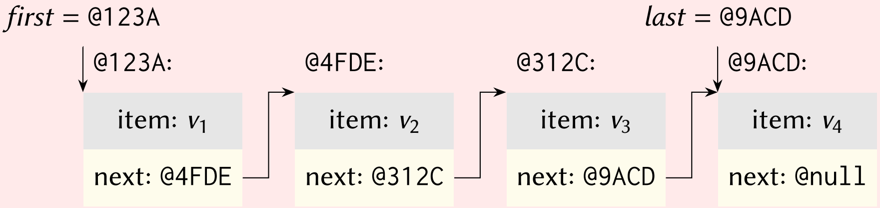

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AppendNode(L, @4FDE, w).

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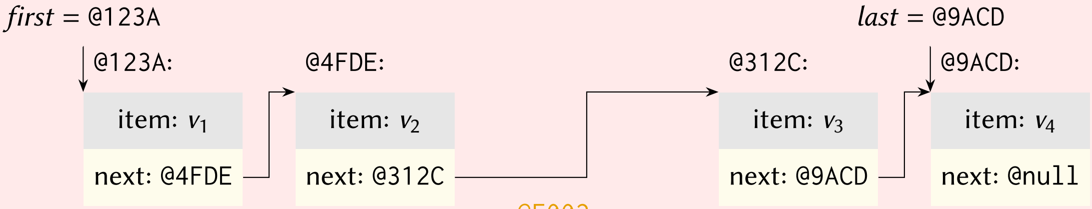

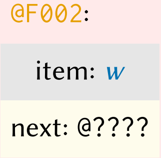

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first = `@123A` *last* = `@9ACD`

`@123A:` item: v_1 — next: `@4FDE`

`@4FDE:` item: v_2 — next: `@312C`

`@312C:` item: v_3 — next: `@9ACD`

`@9ACD:` item: v_4 — next: `@null`

`@F002:` item: w — next: `@312C`

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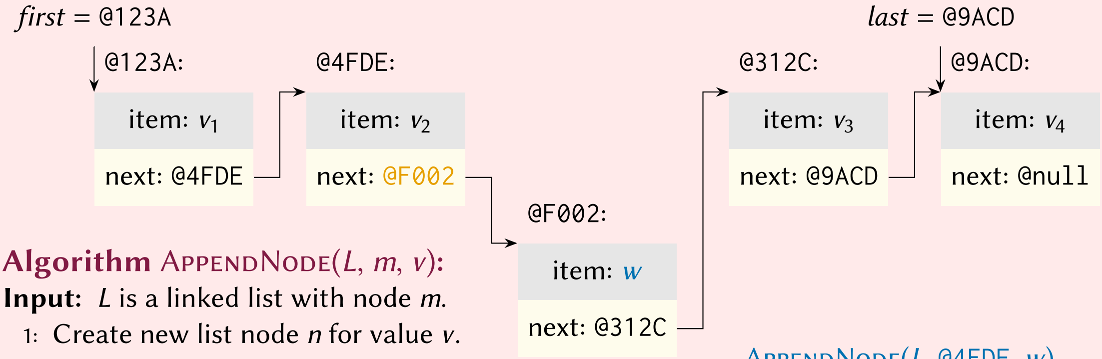

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Removing elements from the front or after a given node is similar:

- PopFront(L) undoes PushFront.
- RemoveNode(L, w) undoes AppendNode (removes a node after node w).

Singly linked lists

Linked Lists: a data structure that can hold a sequence of values, each stored in a *list node*.

- *first* Pointer to the *first list node*.
- *last* Pointer to the *last list node* (optional, for adding values to the end).

Each value in a linked list is held by a *list node*:

- *item* The value held by the list node.
- *next* A pointer to the next list node in the linked list, if any.

Removing elements from the front or after a given node is similar:

- PopFront(L) undoes PushFront.
- RemoveNode(L, w) undoes AppendNode (removes a node after node w).

Make sure to free the memory associated with nodes.

In C++: use either `std::unique_ptr` or `std::shared_ptr` to free nodes *for* you.

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- *Drawback*: Low performance due to *pointer-chasing*, memory overhead of list nodes.

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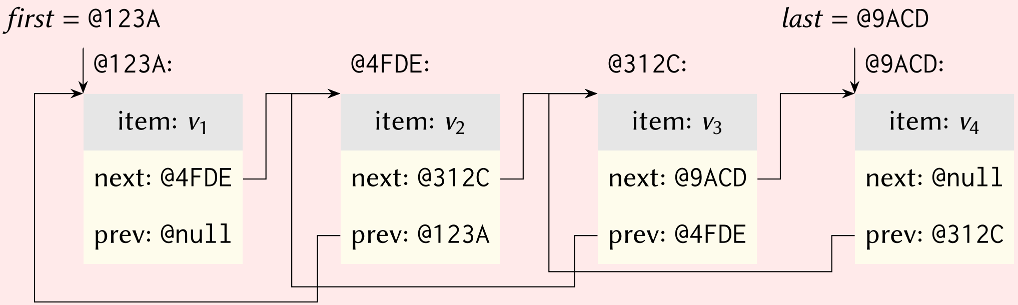

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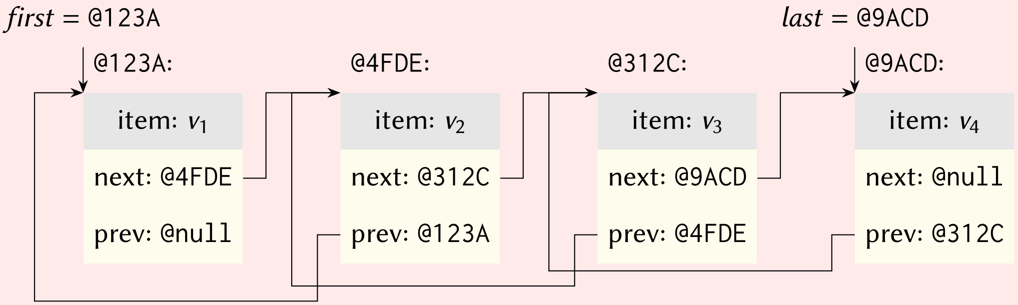

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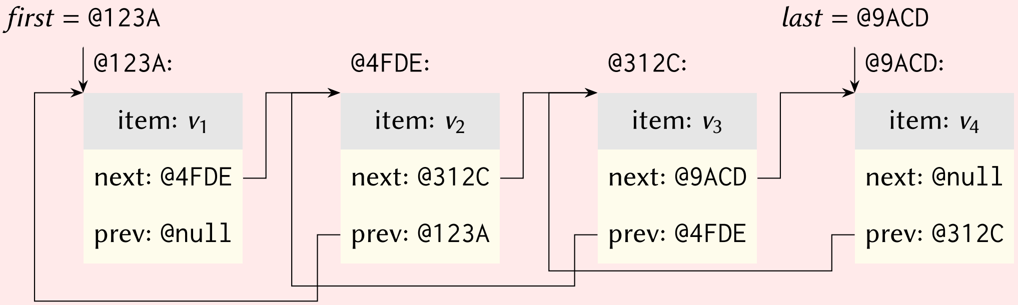

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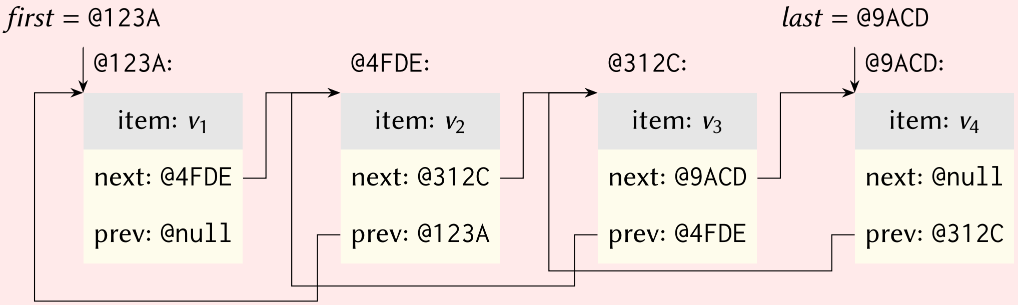

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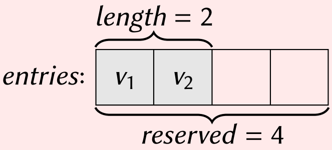

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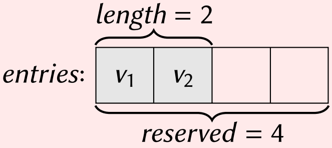

Algorithm $\textsc{PushBack}(D, v)$**:**

Input: D is a dynamic array.

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if D.reserved = D.length then
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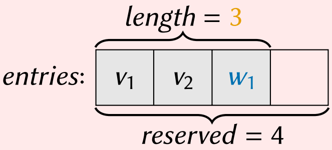

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PushBack(D, w_1); PushBack(D, w_2); PushBack(D, w_3); PushBack(D, w_4).

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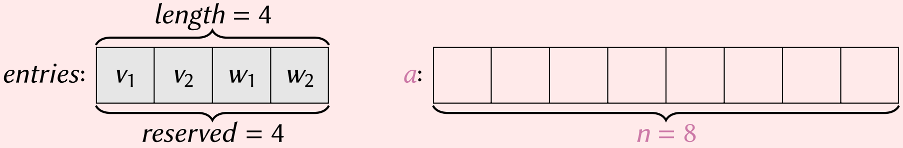

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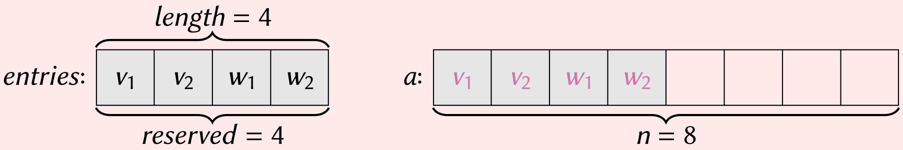

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?

a: $\underbrace{\begin{array}{|c|c|c|c|c|c|c|c|}\hline v_1 & v_2 & w_1 & w_2 & & & & \\ \hline\end{array}}_{n = 8}$

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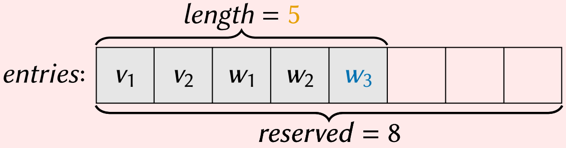

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Can we provide a better analysis?

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Definition (Amortized complexity)

Complexity of a sequence of operations averaged out over all operations performed.

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Amortized complexity is not average case complexity!

Average case complexity looks at the average cost of a *single* operation.

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After M PushBacks on D: $D.length = M$ with $2^j < M \leq D.reserved = 2^{j+1}$.

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- A call to InternalResize(D) with $D.length = i$ costs $c \cdot i + d$ with c, d constants.
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Intermezzo: Complexity of PushBack

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To support PopBack efficiently: do not *shrink* too soon:
either *never shrink* or shrink *if requested* or when $4 \cdot D.length < D.reserved$.

Dynamic arrays

Dynamic Array: a data structure that can hold an *array* of values that is resizes upon need.

Removing elements from the the back is similar:
PopBack(D) undoes PushBack (typically without shrinking).

Arbitrary inserts at position i, $i \neq D.length$, is costly:
Requires one to copy over by one position all values at-or-after position i.

What about the complexity?

PushBack(D, v) is either $\Theta(1)$ or $\Theta(D.reserved)$ (if InternalResize is called).

Can we provide a better analysis?

Amortized complexity of PushBack: $\Theta(1)$.

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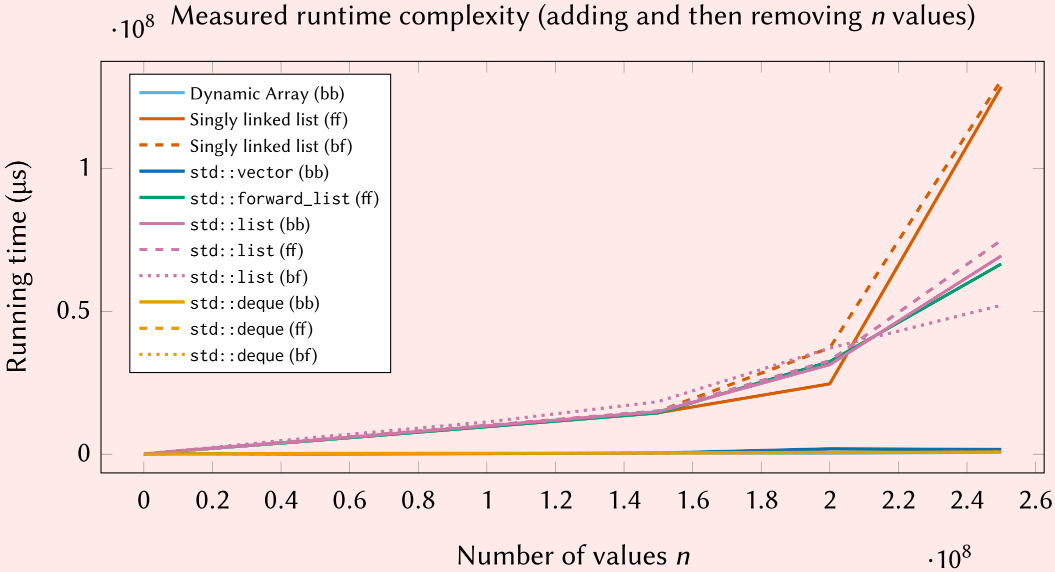
- Queue modifications in $\Theta(D.length)$.
- Support *random access* efficiently.
- *Drawback*: sometimes expensive *resizes*, no efficient queue operations.

A summary of elementary containers

Data structure	Supports ADT: Queue	Stack	Dequeue	Random Access	Memory Usage
Ring Buffer	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	Always NT
Singly linked list	$\Theta(1)$	$\Theta(1)$			$T + P + M$ per value
Doubly linked list	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$		$T + 2P + M$ per value
Dynamic array		$\Theta(1)$ (amortized)		$\Theta(1)$	$\leq M + 2T$ $\leq 2M + 3T$ (during resize)

T is the size of a value, P is the size of a pointer, M is the overhead per memory allocation.

Comparing common containers



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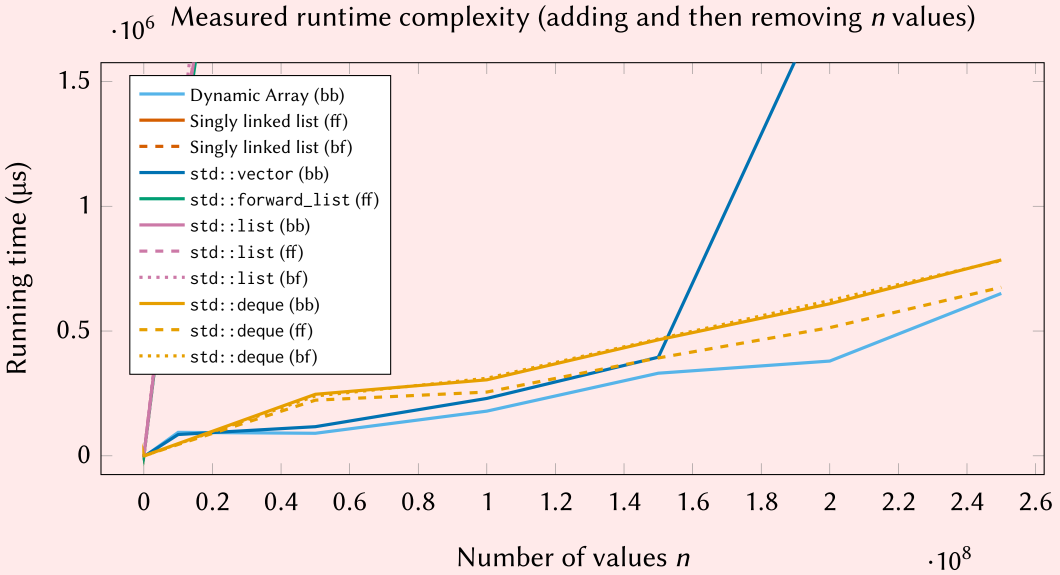

Elementary containers in practice

Data Collection or Structure	C++	Java
Ring Buffer		`java.util.ArrayDeque`
Singly Linked List	`std::forward_list`	
Doubly Linked List	`std::list`	`java.util.LinkedList`
Dynamic Array	`std::vector`	`java.util.ArrayList`
Other	`std::deque`	
Stack	`std::stack`	Use `ArrayDeque` or `ArrayList`
Queue	`std::queue`	Use `ArrayDeque`

Java provides `java.util.Vector` and `Stack` and `Queue` on top of `Vector`. These are ancient and their usage is *not recommended*. Use `ArrayList` or `ArrayDeque` instead!