Graphs

SFWRENG 2CO3: Data Structures and Algorithms

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The graph data model

A graph consists of *nodes* and *edges*:

Nodes denote pieces of information;

Edges denote relationships between these pieces.

Given a graph data set, one can often *derive* other information or relationships.

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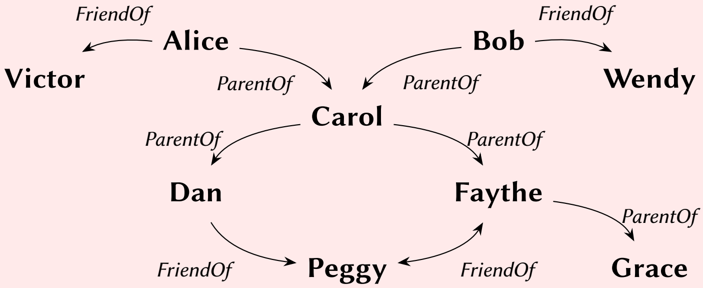
Most data sources can be modeled as graphs, e.g., “Big Data”.
Standard graph algorithms can be used to solve many *different* problems.

Example: Social networks

Source: Hellings et al., 2021.



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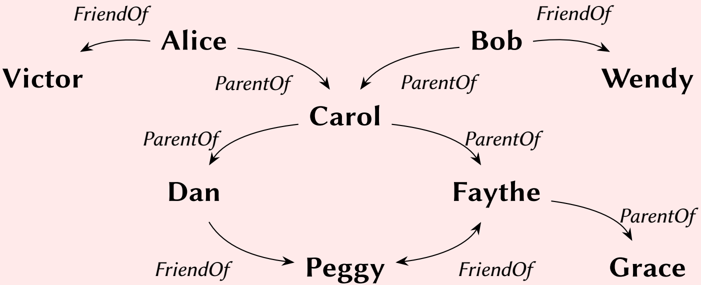

Nodes People.

Edges Relationships between them.

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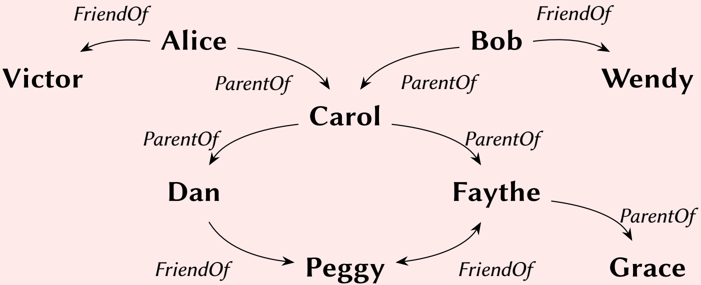

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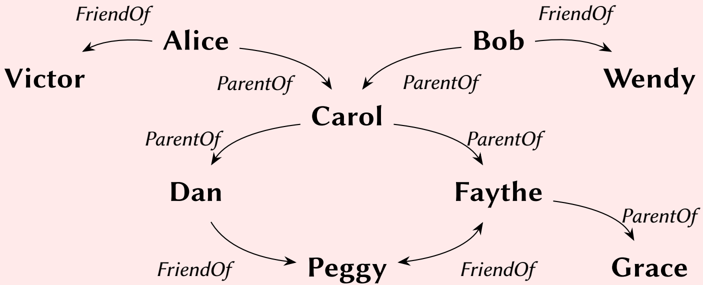

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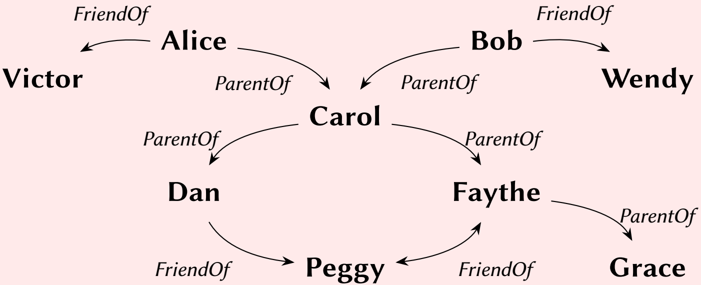

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How can one contact someone else via a friend-of-a-friend? → A shortest path!

Example: Class hierarchy (trees)

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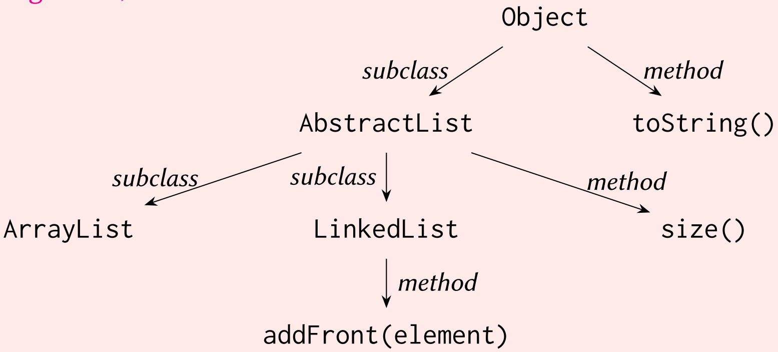

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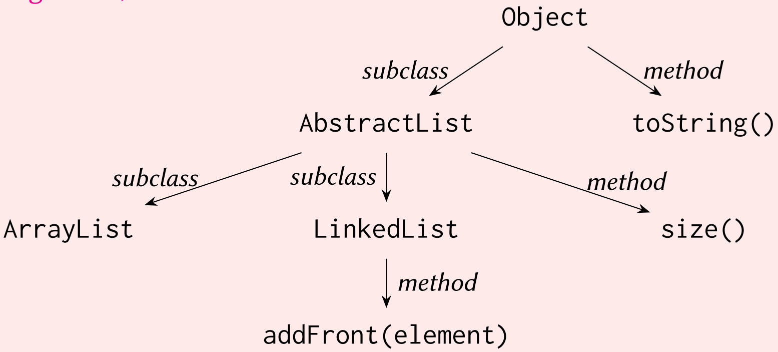

Nodes Names (classes, methods).

Edges Membership (subclass, method).

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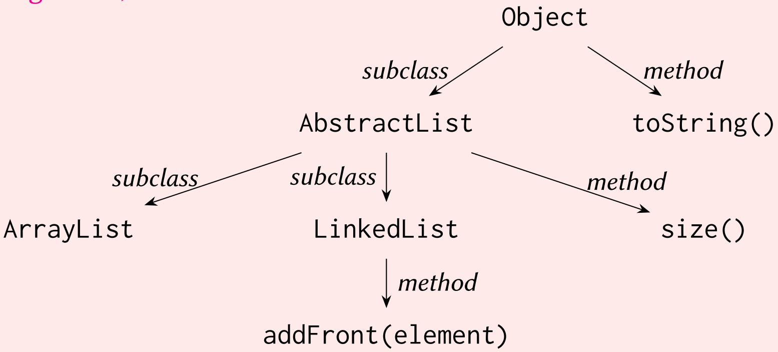

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Question: does `LinkedList` have a method `toString()`?

Example: Rail Network

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The shortest path problem

Which route should a train take to connect stations A and B (with minimal travel time)?

Example: Air traffic data

Source: *On-Time : Reporting Carrier On-Time Performance* at Bureau of Transportation Statistics.

OP_CARRIER	TAIL_NUM	ORIGIN	DEST	DEP_TIME	ARR_DELAY	DISTANCE
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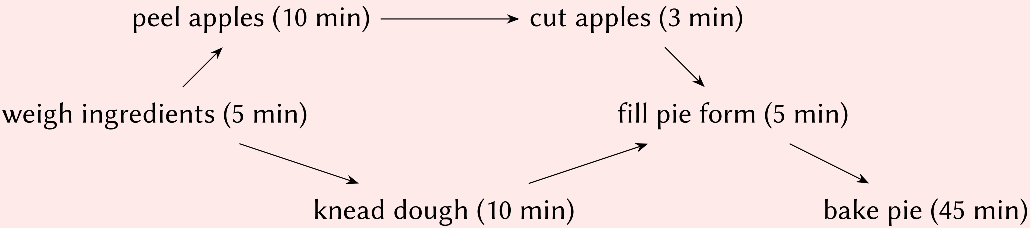
This data has a time component: a *temporal graph*.

Example: Schedules with dependencies (DAGs)



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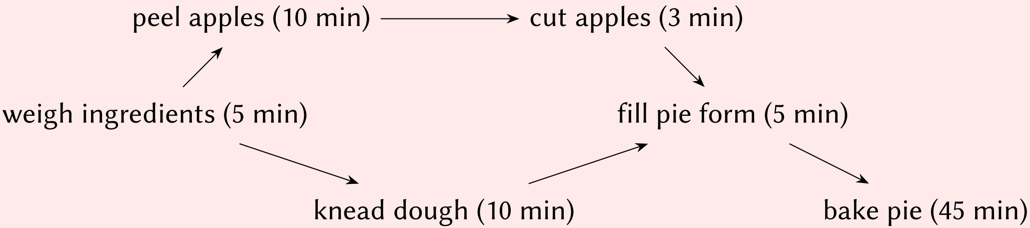

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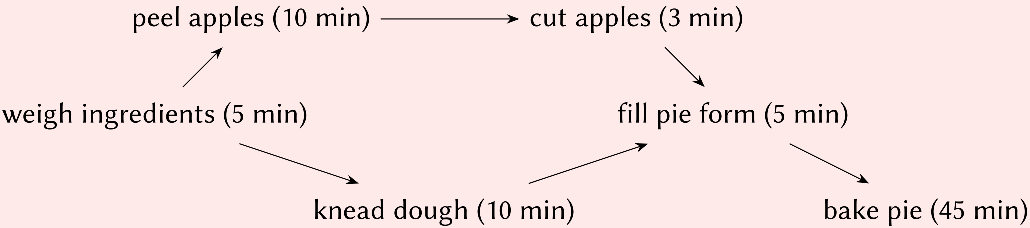

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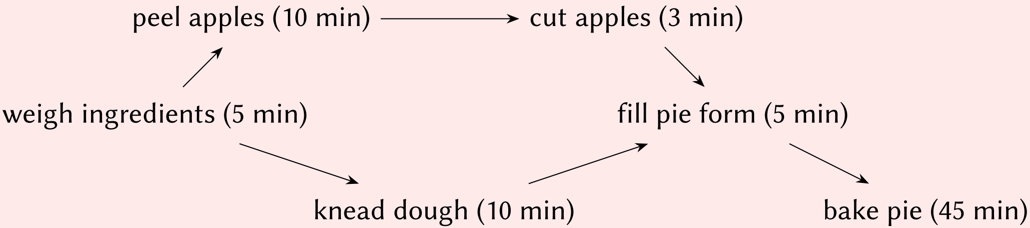

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How fast can a group bake a pie? $\longrightarrow$ A *longest* path problem
(that we can turn into a *shortest* path problem).

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We will see examples of this in the lectures and assignments!

Selected topics on graphs

- Formalization.
- Data structures to represent graphs.
- Traversing graphs:
  Reachability, finding cycles, shortest paths (without weights), topological sort, ...
- Minimum spanning trees.
- Finding shortest-paths (with weights).

Undirected graphs

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  Undirected: if $(v, w) \in \mathcal{E}$, then also $(w, v) \in \mathcal{E}$!

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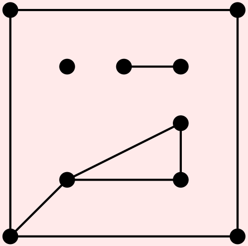

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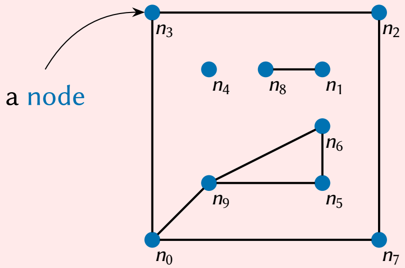

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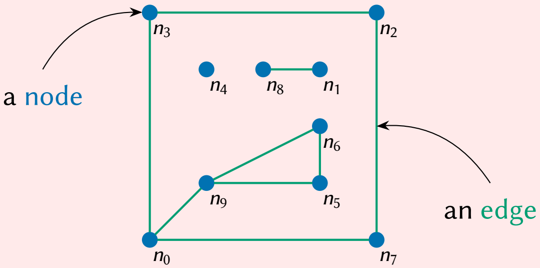

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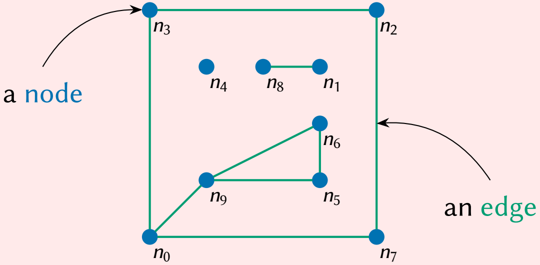
Nodes have unique identities, e.g., they are assigned unique numbers.



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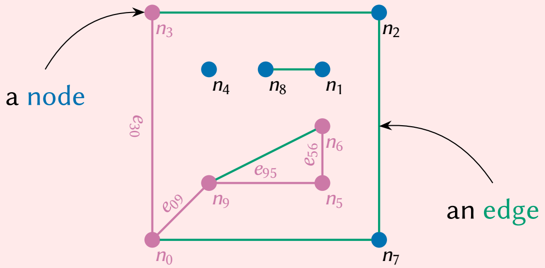
Example: $n_3\, e_{30}\, n_0\, e_{09}\, n_9\, e_{95}\, n_5\, e_{56}\, n_6$.



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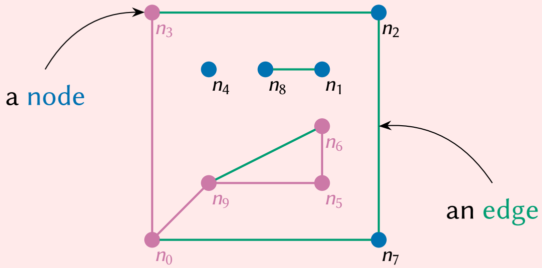

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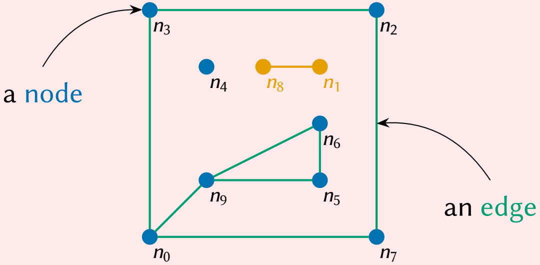
Two nodes are *connected* if there is a path between them.

Connected component: maximal subgraph in which all node pairs are connected.



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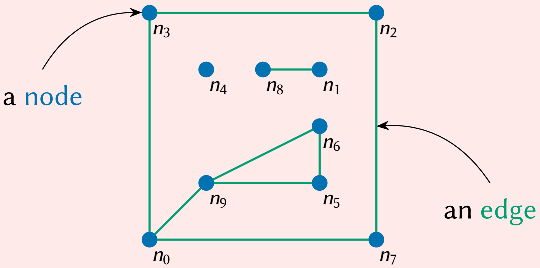
A graph is *connected* if all node pairs are connected.

This graph is *not* connected: there are three disconnected components!



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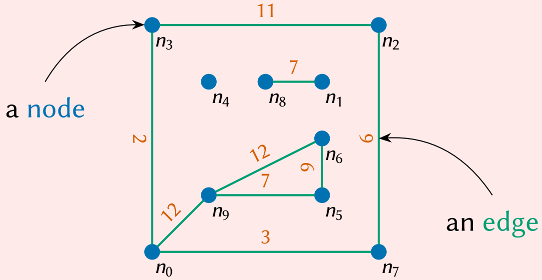
In a *weighted undirected graph*, each edge has a weight.
Typically modeled via a *weight function* *weight*, e.g., $\mathit{weight} : \mathcal{E} \to \mathbb{N}$.



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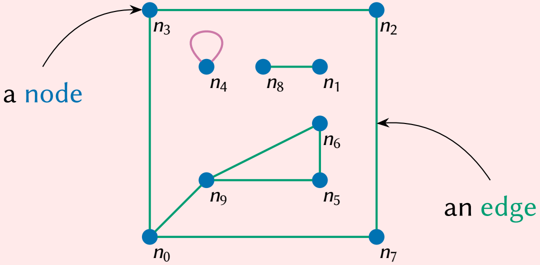
We can have edges from nodes to themselves: self-loops.
(we will mostly ignore self-loops).



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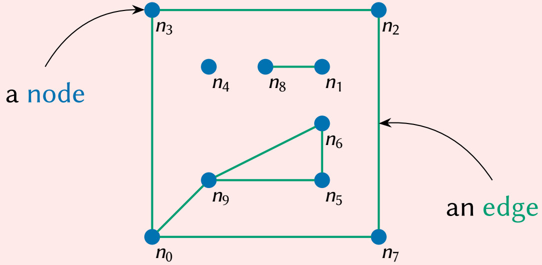

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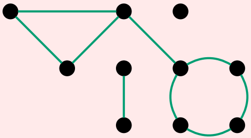

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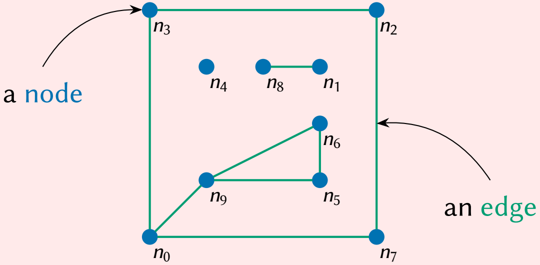

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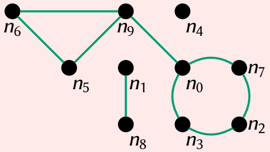

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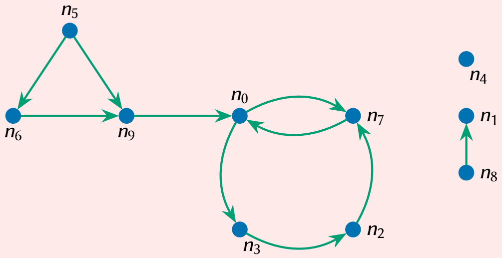

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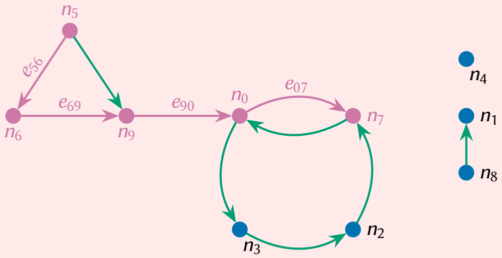

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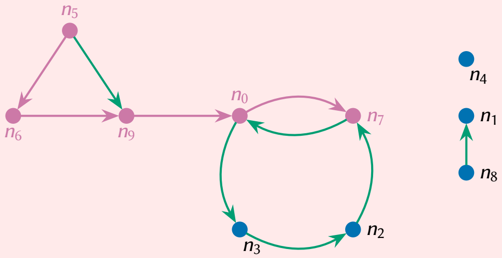

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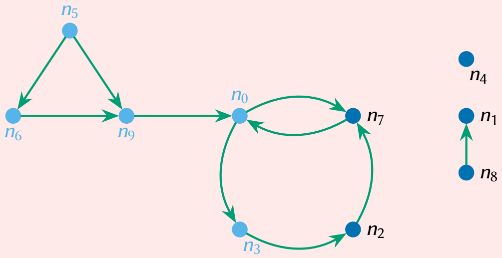
$n_3 n_0 n_9 n_5 n_6$ does not follow direction $\longrightarrow$ *not* a path!



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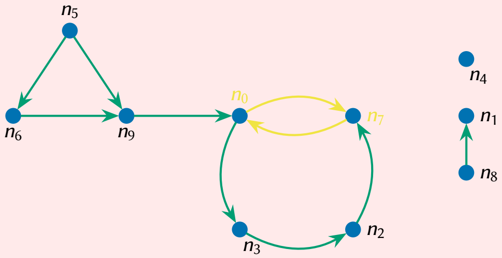
A *directed graph* is a pair $(\mathcal{N}, \mathcal{E})$ with

- $\mathcal{N}$ a set of *nodes* (or *vertices*); and
- $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ a set of *edges* that consist of *node pairs*.

A *cycle* is a path with at-least one edge from a node to itself.

Example: the cycles $n_0 n_7$ and $n_7 n_0$.



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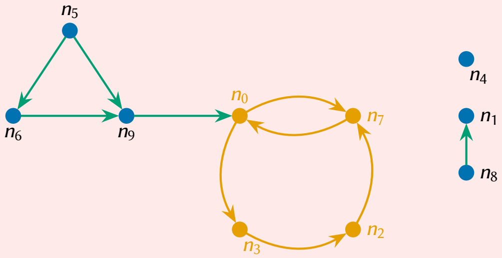
Two nodes are *strongly connected* if there is a path between them.

Strongly …component: maximal subgraph in which all node pairs are strongly connected.



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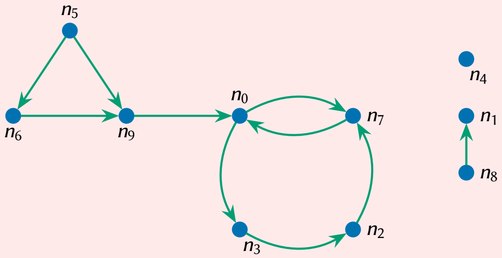
A graph is *strongly connected* if all node pairs are strongly connected.
This graph is *not* strongly connected: e.g., no paths toward n_4.



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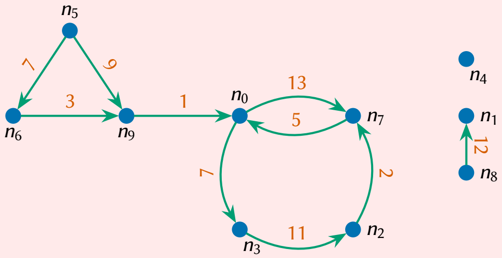
In a *weighted directed graph*, each edge has a weight.
Typically modeled via a *weight function* *weight*, e.g., $\mathit{weight} : \mathcal{E} \to \mathbb{N}$.



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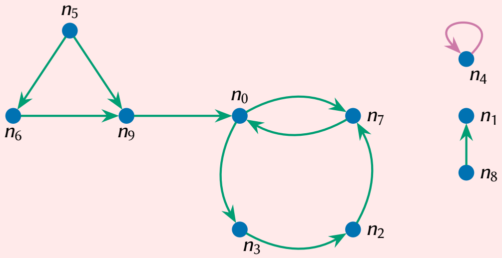
We can have edges from nodes to themselves: self-loops.
(we will mostly ignore self-loops).



Implementing graphs

Consider a directed or undirected graph,
possibly with a weight function.

Which basic operations do we want?

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- Adding and removing edges?

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Which basic operations do we want?

- Adding and removing nodes?
- Adding and removing edges?
- Check whether an edge exists between a node pair?
- Iterate over all (incoming and outgoing) edges of a node?
- Given an edge, check or change the weight?

The matrix representation

Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be a directed graph.
Assume each node $n \in \mathcal{N}$ has a unique identifier $\text{id}(n)$ with $0 \leq \text{id}(n) < |N|$.

Matrix representation

Let M be a $|\mathcal{N}| \times |\mathcal{N}|$-*matrix* ($M$ is a *two-dimensional array*).
For every pair of nodes (m, n), set $M[\text{id}(m), \text{id}(n)] := (m, n) \in \mathcal{E}$.

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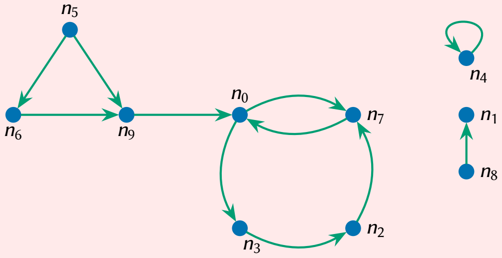

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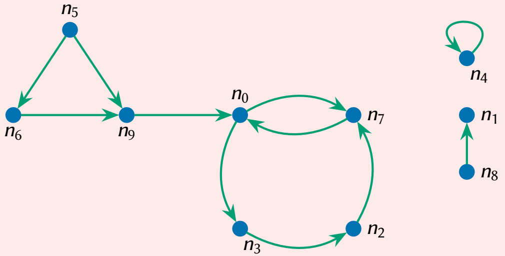

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	1	0	0	0	1	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	1	0	0
3	0	0	1	0	0	0	0	0	0	0
4	0	0	0	0	1	0	0	0	0	0
5	0	0	0	0	0	0	1	0	0	1
6	0	0	0	0	0	0	0	0	0	1
7	1	0	0	0	0	0	0	0	0	0
8	0	1	0	0	0	0	0	0	0	0
9	1	0	0	0	0	0	0	0	0	0

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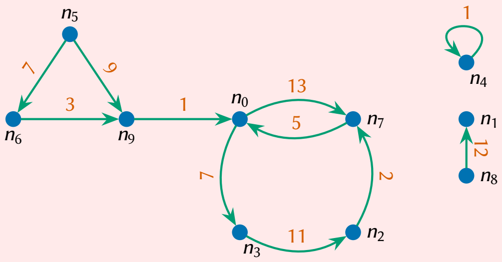
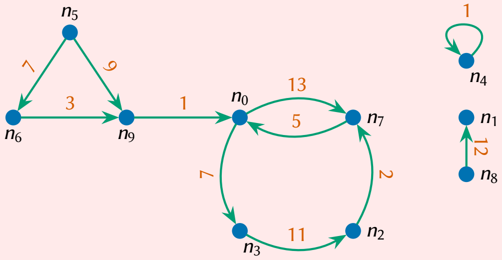


	0	1	2	3	4	5	6	7	8	9
0	*	*	*	7	*	*	*	13	*	*
1	*	*	*	*	*	*	*	*	*	*
2	*	*	*	*	*	*	*	2	*	*
3	*	*	11	*	*	*	*	*	*	*
4	*	*	*	*	1	*	*	*	*	*
5	*	*	*	*	*	*	7	*	*	9
6	*	*	*	*	*	*	*	*	*	3
7	5	*	*	*	*	*	*	*	*	*
8	*	12	*	*	*	*	*	*	*	*
9	1	*	*	*	*	*	*	*	*	*

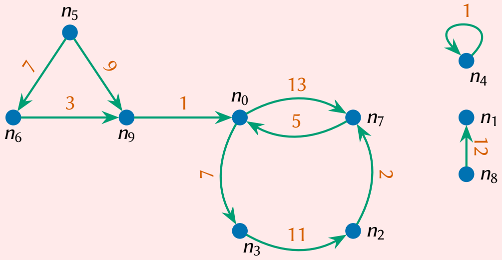
The matrix representation



	0	1	2	3	4	5	6	7	8	9
0	*	*	*	7	*	*	*	13	*	*
1	*	*	*	*	*	*	*	*	*	*
2	*	*	*	*	*	*	*	2	*	*
3	*	*	11	*	*	*	*	*	*	*
4	*	*	*	*	1	*	*	*	*	*
5	*	*	*	*	*	*	7	*	*	9
6	*	*	*	*	*	*	*	*	*	3
7	5	*	*	*	*	*	*	*	*	*
8	*	12	*	*	*	*	*	*	*	*
9	1	*	*	*	*	*	*	*	*	*

- Adding and removing nodes?
- Adding and removing edges (n, m)?
- Check whether an edge (n, m) exists?
- Iterate over all incoming edges of node n?
- Iterate over all outgoing edges of node n?
- Check or change the weight of (n, m)?

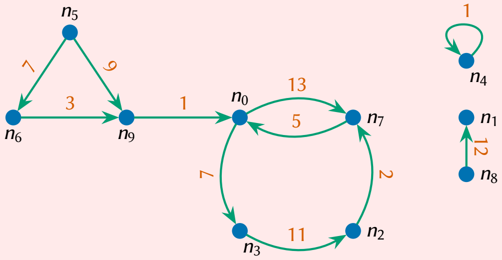
The matrix representation



	0	1	2	3	4	5	6	7	8	9
0	*	*	*	7	*	*	*	13	*	*
1	*	*	*	*	*	*	*	*	*	*
2	*	*	*	*	*	*	*	2	*	*
3	*	*	11	*	*	*	*	*	*	*
4	*	*	*	*	1	*	*	*	*	*
5	*	*	*	*	*	*	7	*	*	9
6	*	*	*	*	*	*	*	*	*	3
7	5	*	*	*	*	*	*	*	*	*
8	*	12	*	*	*	*	*	*	*	*
9	1	*	*	*	*	*	*	*	*	*

- Adding and removing nodes?
- Adding and removing edges (n, m)? $\longrightarrow \Theta(1)$
- Check whether an edge (n, m) exists? $\longrightarrow \Theta(1)$
- Iterate over all incoming edges of node n?
- Iterate over all outgoing edges of node n?
- Check or change the weight of (n, m)? $\longrightarrow \Theta(1)$

The matrix representation



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4	*	*	*	*	1	*	*	*	*	*
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6	*	*	*	*	*	*	*	*	*	3
7	5	*	*	*	*	*	*	*	*	*
8	*	12	*	*	*	*	*	*	*	*
9	1	*	*	*	*	*	*	*	*	*

- Adding and removing nodes? $\longrightarrow \Theta\left(|\mathcal{N}|^2\right)$ (copy to new matrix).
- Adding and removing edges (n, m)? $\longrightarrow \Theta\left(1\right)$
- Check whether an edge (n, m) exists? $\longrightarrow \Theta\left(1\right)$
- Iterate over all incoming edges of node n? $\longrightarrow \Theta\left(|\mathcal{N}|\right)$ (scan a column)
- Iterate over all outgoing edges of node n? $\longrightarrow \Theta\left(|\mathcal{N}|\right)$ (scan a row)
- Check or change the weight of (n, m)? $\longrightarrow \Theta\left(1\right)$

The adjacency list representation

Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be a directed graph.
Assume each node $n \in \mathcal{N}$ has a unique identifier $\text{id}(n)$ with $0 \leq \text{id}(n) < |\mathcal{N}|$.

Adjacency list representation

Let $A[0 \dots |\mathcal{N}|)$ be an array of *bags*.
For every edge $(m, n) \in \mathcal{E}$, Add (m, n) to the bag $A[\text{id}(m)]$.

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- The *standard* adjacency list stores *outgoing* edges.
  If needed, one can also store *incoming edges* or *both*.
- $A[i]$ is a *bag*, e.g., linked list, dynamic array, search tree, hash table, ...
- A can be a *dynamic array* to support adding nodes efficiently.
- A can be a *dictionary* mapping nodes onto their adjacency lists.
  Useful when nodes do not have identifiers, not all nodes have edges, ...

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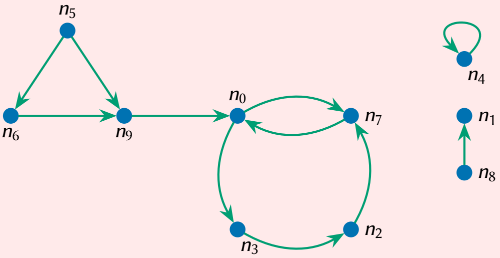

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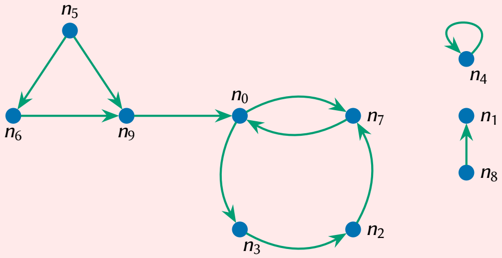

0	$[(n_0, n_3), (n_0, n_7)]$
1	$[]$
2	$[(n_2, n_7)]$
3	$[(n_3, n_2)]$
4	$[(n_4, n_4)]$
5	$[(n_5, n_6), (n_5, n_9)]$
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8	$[(n_8, n_1)]$
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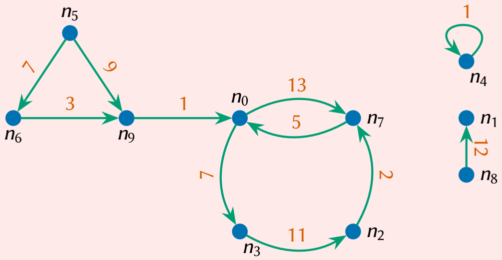

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2	$[(n_2, n_7) : 2]$
3	$[(n_3, n_2) : 11]$
4	$[(n_4, n_4) : 1]$
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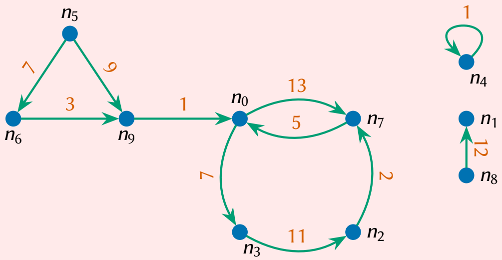

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- Adding and removing nodes?
- Adding and removing edges (n, m)?
- Check whether an edge (n, m) exists?
- Iterate over all *incoming* edges of node n?
- Iterate over all *outgoing* edges of node n?
- Check or change the weight of (n, m)?

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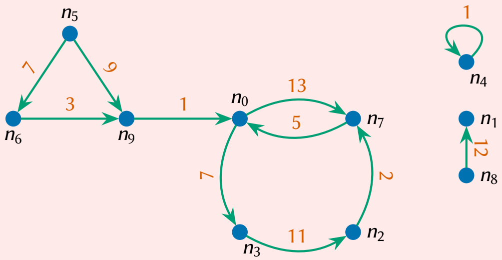

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- Check or change the weight of (n, m)? $\longrightarrow \Theta(1)$

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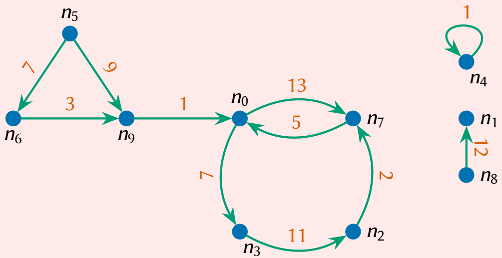

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- Adding and removing nodes? $\rightarrow \Theta(|\mathcal{N}|)$ (copy array).
- Adding and removing edges (n, m)? $\rightarrow \Theta(|\mathcal{N}|)$ (adding to bag).
- Check whether an edge (n, m) exists? $\rightarrow \Theta(|\mathcal{N}|)$ (searching bag)
- Iterate over all *incoming* edges of node n? $\rightarrow \Theta(|\mathcal{E}|)$ (scan all bags)
- Iterate over all *outgoing* edges of node n? $\rightarrow \Theta(|\mathcal{N}|)$ (scan a bag)
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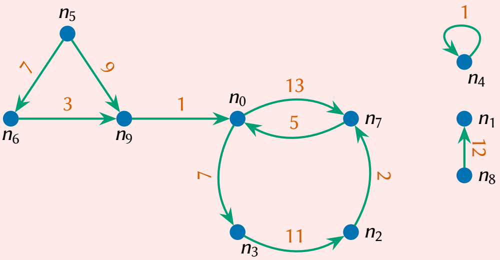
The adjacency list representation



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9	$[(n_9, n_0) : 1]$

Let $\text{out}(n) = \{(n, m) \in \mathcal{E}\}$ be all *outgoing* edges of node n.

- Adding and removing nodes? $\longrightarrow \Theta(|\mathcal{N}|)$ (copy array).
- Adding and removing edges (n, m)? $\longrightarrow \Theta(|\text{out}(n)|)$ (adding to bag).
- Check whether an edge (n, m) exists? $\longrightarrow \Theta(|\text{out}(n)|)$ (searching bag)
- Iterate over all *incoming* edges of node n? $\longrightarrow \Theta(|\mathcal{E}|)$ (scan all bags)
- Iterate over all *outgoing* edges of node n? $\longrightarrow \Theta(|\text{out}(n)|)$ (scan a bag)
- Check or change the weight of (n, m)? $\longrightarrow \Theta(1)$

A comparison of representations

Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be a directed graph.

Dense graph graph $\mathcal{G}$ is *dense* if $|\mathcal{E}| = \Theta\left(|\mathcal{N}|^2\right)$.

Sparse graph graph $\mathcal{G}$ is *spase* if $|\mathcal{E}| = \Theta\left(|\mathcal{N}|\right)$.

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Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be a directed graph.

Dense graph: graph $\mathcal{G}$ is *dense* if $|\mathcal{E}| = \Theta\left(|\mathcal{N}|^2\right)$. $\longrightarrow$ most node pairs are edges!

Sparse graph: graph $\mathcal{G}$ is *spase* if $|\mathcal{E}| = \Theta\left(|\mathcal{N}|\right)$. $\longrightarrow$ most node pairs are *not* edges!

	Matrix		Adjacency List	
	Sparse	Dense	Sparse	Dense
Memory usage	$\Theta\left(\lvert\mathcal{N}\rvert^2\right)$		$\Theta\left(\lvert\mathcal{N}\rvert + \lvert\mathcal{E}\rvert\right)$	
Adding nodes	$\Theta\left(\lvert\mathcal{N}\rvert^2\right)$		$\Theta\left(\lvert\mathcal{N}\rvert\right)$	
Adding edge (n, m)	$\Theta\left(1\right)$		$\Theta\left(\lvert\text{out}(n)\rvert\right)$	
Checking edge (n, m)	$\Theta\left(1\right)$		$\Theta\left(\lvert\text{out}(n)\rvert\right)$	
Incoming edges of n	$\Theta\left(\lvert\mathcal{N}\rvert\right)$		$\Theta\left(\lvert\mathcal{E}\rvert\right)$	
Outgoing edges of n	$\Theta\left(\lvert\mathcal{N}\rvert\right)$		$\Theta\left(\lvert\text{out}(n)\rvert\right)$	
Weight of edge (n, m)	$\Theta\left(1\right)$		$\Theta\left(\lvert\text{out}(n)\rvert\right)$	

A comparison of representations

Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be a directed graph.

Dense graph graph $\mathcal{G}$ is *dense* if $|\mathcal{E}| = \Theta\left(|\mathcal{N}|^2\right)$. $\rightarrow$ most node pairs are edges!

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	Matrix		Adjacency List	
	Sparse	Dense	Sparse	Dense
Memory usage	$\Theta\left(\lvert\mathcal{N}\rvert^2\right)$	$\Theta\left(\lvert\mathcal{N}\rvert^2\right)$	$\Theta\left(\lvert\mathcal{N}\rvert + \lvert\mathcal{E}\rvert\right)$	$\Theta\left(\lvert\mathcal{N}\rvert^2\right)$
Adding nodes	$\Theta\left(\lvert\mathcal{N}\rvert^2\right)$	$\Theta\left(\lvert\mathcal{N}\rvert^2\right)$	$\Theta\left(\lvert\mathcal{N}\rvert\right)$	$\Theta\left(\lvert\mathcal{N}\rvert^2\right)$
Adding edge (n, m)	$\Theta(1)$	$\Theta(1)$	$\Theta\left(\lvert\text{out}(n)\rvert\right)$	$\Theta\left(\lvert\text{out}(n)\rvert\right)$
Checking edge (n, m)	$\Theta(1)$	$\Theta(1)$	$\Theta\left(\lvert\text{out}(n)\rvert\right)$	$\Theta\left(\lvert\text{out}(n)\rvert\right)$
Incoming edges of n	$\Theta\left(\lvert\mathcal{N}\rvert\right)$	$\Theta\left(\lvert\mathcal{N}\rvert\right)$	$\Theta\left(\lvert\mathcal{E}\rvert\right)$	$\Theta\left(\lvert\mathcal{N}\rvert^2\right)$
Outgoing edges of n	$\Theta\left(\lvert\mathcal{N}\rvert\right)$	$\Theta\left(\lvert\mathcal{N}\rvert\right)$	$\Theta\left(\lvert\text{out}(n)\rvert\right)$	$\Theta\left(\lvert\text{out}(n)\rvert\right)$
Weight of edge (n, m)	$\Theta(1)$	$\Theta(1)$	$\Theta\left(\lvert\text{out}(n)\rvert\right)$	$\Theta\left(\lvert\text{out}(n)\rvert\right)$

A comparison of representations

Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be a directed graph.

Dense graph graph $\mathcal{G}$ is *dense* if $|\mathcal{E}| = \Theta\left(|\mathcal{N}|^2\right)$. $\rightarrow$ most node pairs are edges!

Sparse graph graph $\mathcal{G}$ is *spase* if $|\mathcal{E}| = \Theta\left(|\mathcal{N}|\right)$. $\rightarrow$ most node pairs are *not* edges!

Which representation is the best?

- Sparse graphs?
- Dense graphs?
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E.g., graph operations in terms of *matrices* are easier to implement on GPUs.

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Many alternatives exist

- Simply storing the set of *edges* (e.g., as a *relational table* in a database);
- *Compressed matrices* for GPU operations on sparse graphs (e.g., in machine learning);
- ....

Traversing undirected graphs: Depth-first

Algorithm DFS-R($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, *marked*, $n \in \mathcal{N}$):

```
for all (n, m) ∈ ℰ do
    if ¬marked[m] then
        marked[m] := true.
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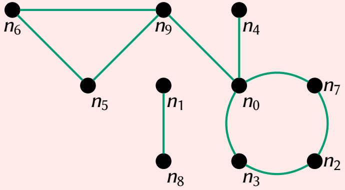

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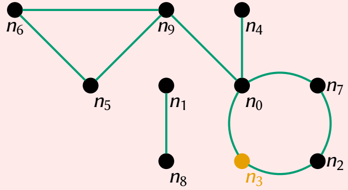

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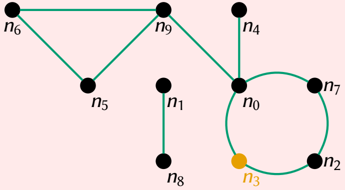
$marked =$

n_0	`false`
n_1	`false`
n_2	`false`
n_3	`true`
n_4	`false`
n_5	`false`
n_6	`false`
n_7	`false`
n_8	`false`
n_9	`false`



Traversing undirected graphs: Depth-first

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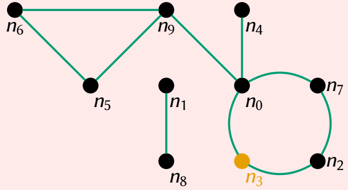

marked =

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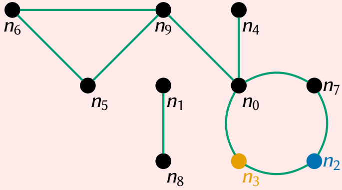

$marked =$

n_0	false
n_1	false
n_2	true
n_3	true
n_4	false
n_5	false
n_6	false
n_7	false
n_8	false
n_9	false

Traversing undirected graphs: Depth-first

Called with $n = n_3$, n_2.

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    DFS-R($\mathcal{G}$, *marked*, m).

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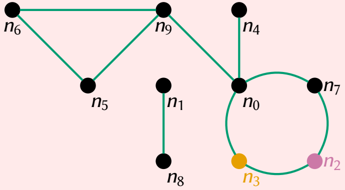

marked =

n_0	false
n_1	false
n_2	true
n_3	true
n_4	false
n_5	false
n_6	false
n_7	false
n_8	false
n_9	false

Traversing undirected graphs: Depth-first

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n_3	true
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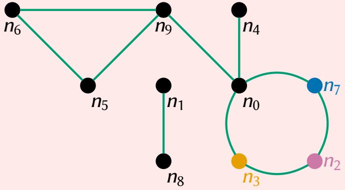

Traversing undirected graphs: Depth-first

Called with $n = n_3, n_2, n_7$.

Algorithm DFS-R($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, *marked*, $n \in \mathcal{N}$):

for all $(n, m) \in \mathcal{E}$ **do**
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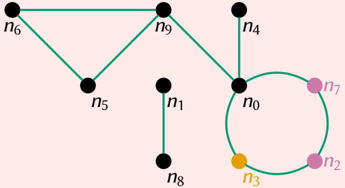

marked =

n_0	`false`
n_1	`false`
n_2	`true`
n_3	`true`
n_4	`false`
n_5	`false`
n_6	`false`
n_7	`true`
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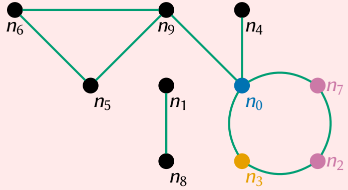
n_0	true
n_1	false
n_2	true
n_3	true
n_4	false
n_5	false
n_6	false
n_7	true
n_8	false
n_9	false



Traversing undirected graphs: Depth-first

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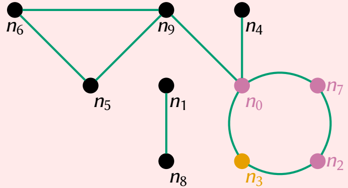

Traversing undirected graphs: Depth-first

Called with $n = n_3, n_2, n_7, n_0$.

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for all (n, m) ∈ E do
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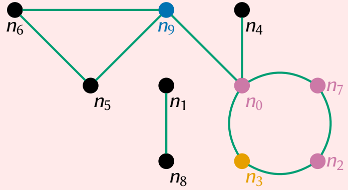

Traversing undirected graphs: Depth-first

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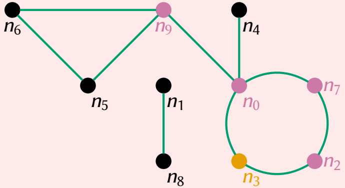

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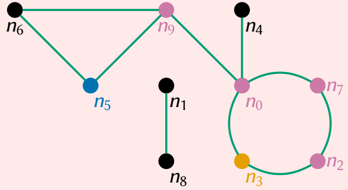

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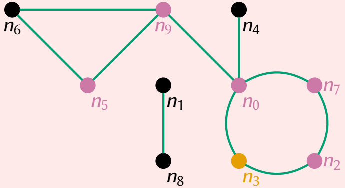

Traversing undirected graphs: Depth-first

Called with $n = n_3, n_2, n_7, n_0, n_9, n_5$.

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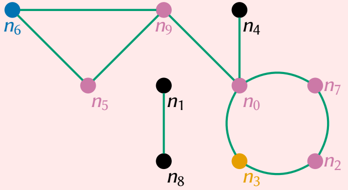

Traversing undirected graphs: Depth-first

Called with $n = n_3, n_2, n_7, n_0, n_9, n_5, n_6$.

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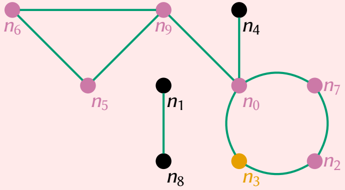

marked =

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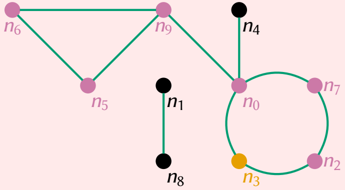
```
marked := {n ↦ (n ≠ s) | n ∈ 𝒩}.
DFS-R(𝒢, marked, s).
```

$$marked = \begin{array}{c|c|} n_0 & \texttt{true} \\ n_1 & \texttt{false} \\ n_2 & \texttt{true} \\ n_3 & \texttt{true} \\ n_4 & \texttt{false} \\ n_5 & \texttt{true} \\ n_6 & \texttt{true} \\ n_7 & \texttt{true} \\ n_8 & \texttt{false} \\ n_9 & \texttt{true} \end{array}$$



Traversing undirected graphs: Depth-first

Called with $n = n_3, n_2, n_7, n_0, n_9, n_5$.

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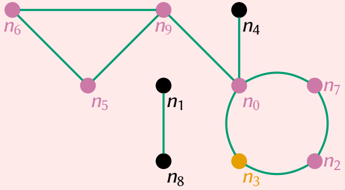
n_8	false
n_9	true



Traversing undirected graphs: Depth-first

Called with $n = n_3, n_2, n_7, n_0, n_9$.

Algorithm DFS-R($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, *marked*, $n \in \mathcal{N}$):

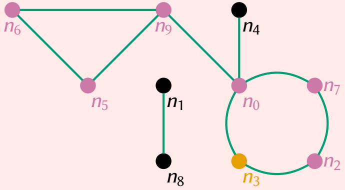
for all $(n, m) \in \mathcal{E}$ **do**
    if $\neg marked[m]$ **then**
        $marked[m]$:= `true`.
        DFS-R($\mathcal{G}$, *marked*, m).

Algorithm DepthFirstR($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, $s \in \mathcal{N}$):

$marked := \{n \mapsto (n \neq s) \mid n \in \mathcal{N}\}$.
DFS-R($\mathcal{G}$, *marked*, s).

marked =

n_0	`true`
n_1	`false`
n_2	`true`
n_3	`true`
n_4	`false`
n_5	`true`
n_6	`true`
n_7	`true`
n_8	`false`
n_9	`true`



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for all $(n, m) \in \mathcal{E}$ **do**
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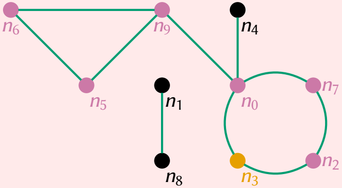

Traversing undirected graphs: Depth-first

Called with $n = n_3, n_2, n_7, n_0, n_4$.

Algorithm DFS-R($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, *marked*, $n \in \mathcal{N}$):

for all $(n, m) \in \mathcal{E}$ **do**
  if $\neg marked[m]$ **then**
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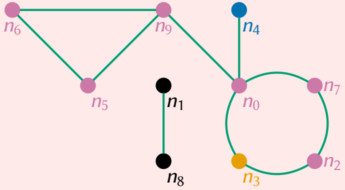

marked =

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n_2	true
n_3	true
n_4	true
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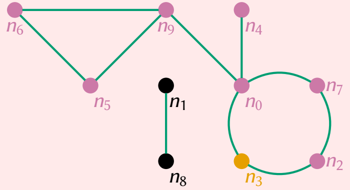

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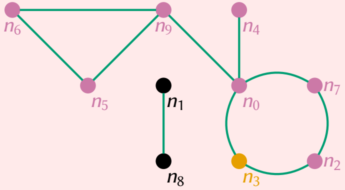

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n_4	true
n_5	true
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n_7	true
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Traversing undirected graphs: Depth-first

Called with $n = n_3, n_2, n_7$.

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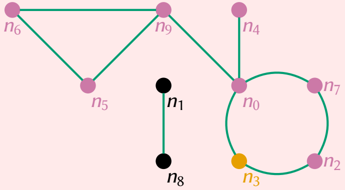

Traversing undirected graphs: Depth-first

Called with $n = n_3, n_2$.

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n_2	true
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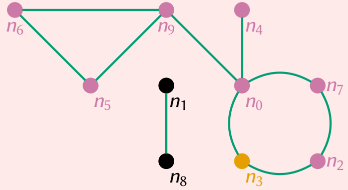

Traversing undirected graphs: Depth-first

Called with $n = n_3$.

Algorithm DFS-R($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, *marked*, $n \in \mathcal{N}$):

```
for all (n, m) ∈ ℰ do
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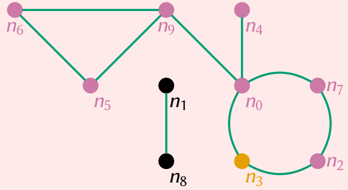

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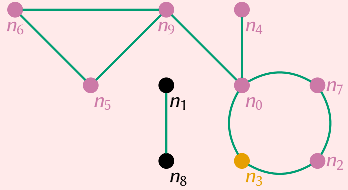

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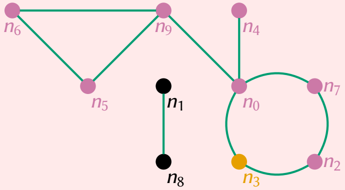
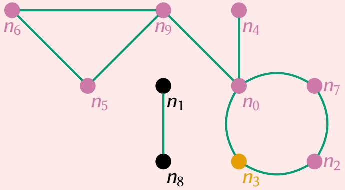


What can we learn from this depth-first search?

Traversing undirected graphs: Depth-first



What can we learn from this depth-first search?

- We found all nodes to which n_3 is *connected* (nodes one can reach from n_3).

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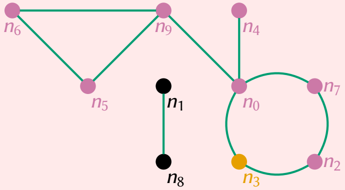

What can we learn from this depth-first search?

- We found all nodes to which n_3 is *connected* (nodes one can reach from n_3).
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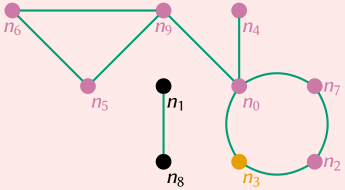
Traversing undirected graphs: Depth-first



What can we learn from this depth-first search?

- We found all nodes to which n_3 is *connected* (nodes one can reach from n_3).
- $\mathcal{G}$ is *not* a connected graph.
- The order of recursive calls was:

$$n = n_3, n_2, n_7, n_0, \begin{cases} n_9, n_5, n_6; \\ n_4. \end{cases}$$

This order provides a path from n_3 to *every* node it is connected to!

Traversing undirected graphs: Depth-first

Algorithm DFS-R($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, *marked*, $n \in \mathcal{N}$):

for all $(n, m) \in \mathcal{E}$ **do**
  if $\neg marked[m]$ **then**
    $marked[m] := \texttt{true}$.
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Algorithm DepthFirstR($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, $s \in \mathcal{N}$):

$marked := \{n \mapsto (n \neq s) \mid n \in \mathcal{N}\}$.
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Complexity

Traversing undirected graphs: Depth-first

Algorithm DFS-R($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, $marked$, $n \in \mathcal{N}$):

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        DFS-R(𝒢, marked, m).
```

Algorithm DepthFirstR($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, $s \in \mathcal{N}$):

```
marked := {n ↦ (n ≠ s) | n ∈ 𝒩}.
DFS-R(𝒢, marked, s).
```

Complexity

- We need $|\mathcal{N}|$ memory for *marked* and the at-most $|\mathcal{N}|$ recursive calls.
- We inspect each node once and traverse each edge once: $\Theta(|\mathcal{N}| + |\mathcal{E}|)$ (if we use the adjacency list representation).

Problem: Connected components

Problem

Given an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.

Provide an algorithm that can find all connected components in $\mathcal{G}$.

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Remark: $\textsc{DepthFirstR}(\mathcal{G}, n)$ will find all nodes in the connected component of n.

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Provide an algorithm that can find all connected components in $\mathcal{G}$.

Solution

Remark: DepthFirstR($\mathcal{G}$, n) will find all nodes in the connected component of n.

Algorithm DFS-CC-R($\mathcal{G}$, cc, $n \in \mathcal{N}$):

```
for all (n, m) ∈ ℰ do
    if cc[m] = unmarked then
        cc[m] := cc[n].
        DFS-CC-R(𝒢, cc, m).
```

Algorithm Components($\mathcal{G}$, $s \in \mathcal{N}$):

```
cc := {n ↦ unmarked}.
for all n ∈ 𝒩 do
    if cc[n] = unmarked then
        cc[n] := n.
        DFS-CC-R(𝒢, cc, n).
```

Problem: Connected components

Problem

Given an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.
Provide an algorithm that can find all connected components in $\mathcal{G}$.

Solution

Remark: DEPTHFIRSTR($\mathcal{G}$, n) will find all nodes in the connected component of n.

Algorithm DFS-CC-R($\mathcal{G}$, cc, $n \in \mathcal{N}$):

for all $(n, m) \in \mathcal{E}$ **do**
  if $cc[m] = \mathit{unmarked}$ **then**
    $cc[m] := cc[n]$.
    DFS-CC-R($\mathcal{G}$, cc, m).

Algorithm COMPONENTS($\mathcal{G}$, $s \in \mathcal{N}$):

$cc := \{n \mapsto \mathit{unmarked}\}$.
for all $n \in \mathcal{N}$ **do**
  if $cc[n] = \mathit{unmarked}$ **then**
    $cc[n] := n$.
    DFS-CC-R($\mathcal{G}$, cc, n).

We inspect each node once and traverse each edge once: $\Theta(|\mathcal{N}| + |\mathcal{E}|)$.

Problem: Two-colorability

Problem

Given an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ in which:

- the nodes $\mathcal{N}$ represent competitors;
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Can we divide the nodes into two teams such that no rivals are in the same team?

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Definition

Graph $\mathcal{G}$ is *bipartite* if we can partition the nodes in two sets such that no two nodes in the same set share an edge.

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The two-colorability problem

Given an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. Find a coloring of the nodes $\mathcal{N}$ (if possible) using two colors such that nodes $(n, m) \in \mathcal{E}$ have different colors.

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The two-colorability problem

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Algorithm DFS-TC-R($\mathcal{G}$, *colors*, $n \in \mathcal{N}$):

for all $(n, m) \in \mathcal{E}$ **do**
    if $colors[m] = 0$ **then**
        $colors[m] := -colors[n]$.
        DFS-TC-R($\mathcal{G}$, *colors*, m).
    else if $colors[m] = colors[n]$ **then**
        This graph is *not* bipartite.

Algorithm TwoColors($\mathcal{G}$):

$colors := \{n \mapsto 0 \mid n \in \mathcal{N}\}$.
for all $n \in \mathcal{N}$ **do**
    if $colors[n] = 0$ **then**
        $colors[n] := 1$.
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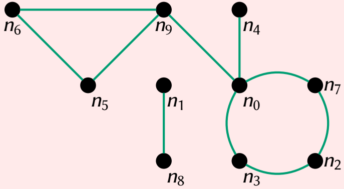

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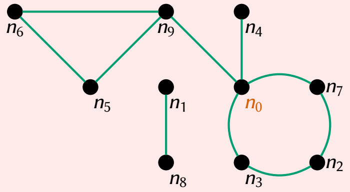

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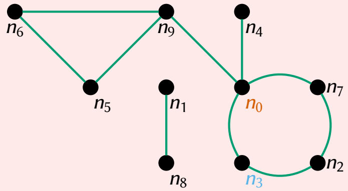

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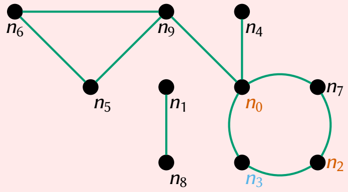

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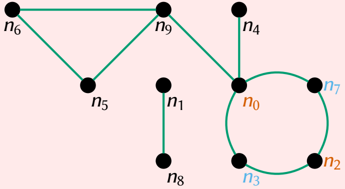

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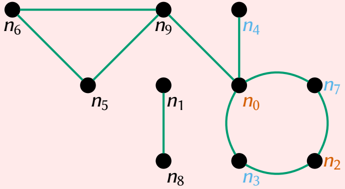

Problem: Two-colorability

Algorithm DFS-TC-R($\mathcal{G}$, *colors*, $n \in \mathcal{N}$):

for all $(n, m) \in \mathcal{E}$ **do**
  if $colors[m] = 0$ **then**
    $colors[m] := -colors[n]$.
    DFS-TC-R($\mathcal{G}$, *colors*, m).
  else if $colors[m] = colors[n]$ **then**
    This graph is *not* bipartite.

Algorithm TwoColors($\mathcal{G}$):

$colors := \{n \mapsto 0 \mid n \in \mathcal{N}\}$.
for all $n \in \mathcal{N}$ **do**
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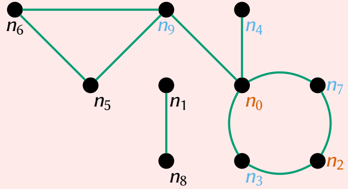
    $colors[n] := 1$.
    DFS-TC-R($\mathcal{G}$, *colors*, n).



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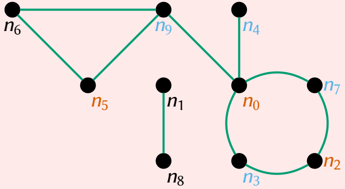

Problem: Two-colorability

Algorithm DFS-TC-R($\mathcal{G}$, *colors*, $n \in \mathcal{N}$):

```
for all (n, m) ∈ ℰ do
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        DFS-TC-R(𝒢, colors, m).
    else if colors[m] = colors[n] then
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Algorithm TwoColors($\mathcal{G}$):

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colors := {n ↦ 0 | n ∈ 𝒩}.
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    if colors[n] = 0 then
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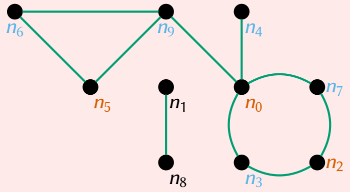

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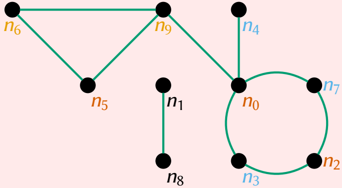

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We inspect each node once and traverse each edge once: $\Theta(|\mathcal{N}| + |\mathcal{E}|)$.

Traversing undirected graphs: Breadth-first

Algorithm BFS($\mathcal{G} = (\mathcal{N}, \mathcal{E}), s \in \mathcal{N}$):

$marked := \{n \mapsto (n \neq s) \mid n \in \mathcal{N}\}$.
Q := a queue holding only s.
while $\neg$Empty(Q) **do**
  $n :=$ Dequeue(Q).
  for all $(n, m) \in \mathcal{E}$ **do**
    if $\neg marked[m]$ **then**
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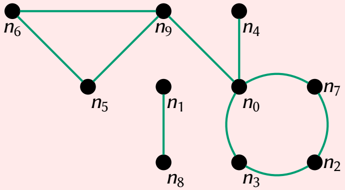

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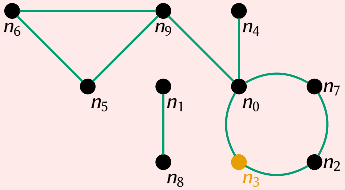

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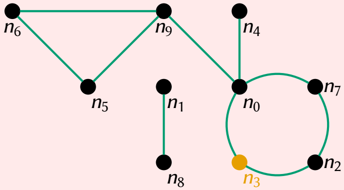
      $marked[m]$:= true.
      Enqueue(S, m).

$marked =$

n_0	false
n_1	false
n_2	false
n_3	true
n_4	false
n_5	false
n_6	false
n_7	false
n_8	false
n_9	false



Traversing undirected graphs: Breadth-first

$Q : [n_3]$.

Algorithm BFS($\mathcal{G} = (\mathcal{N}, \mathcal{E}), s \in \mathcal{N}$):

```
marked := {n ↦ (n ≠ s) | n ∈ 𝒩}.
Q := a queue holding only s.
while ¬Empty(Q) do
    n := Dequeue(Q).
    for all (n, m) ∈ ℰ do
        if ¬marked[m] then
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            Enqueue(S, m).
```
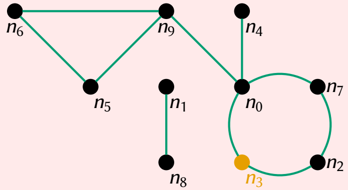

$marked =$

n_0	false
n_1	false
n_2	false
n_3	true
n_4	false
n_5	false
n_6	false
n_7	false
n_8	false
n_9	false



Traversing undirected graphs: Breadth-first

$Q : [n_0, n_2]$, $n = n_3$.

Algorithm BFS($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, $s \in \mathcal{N}$):

$marked := \{n \mapsto (n \neq s) \mid n \in \mathcal{N}\}$.
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  for all $(n, m) \in \mathcal{E}$ **do**
    if $\neg marked[m]$ **then**
      $marked[m] :=$ true.
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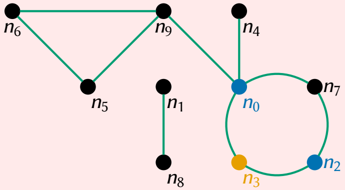
$marked =$

n_0	true
n_1	false
n_2	true
n_3	true
n_4	false
n_5	false
n_6	false
n_7	false
n_8	false
n_9	false



Traversing undirected graphs: Breadth-first

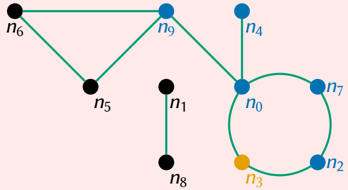
$Q : [n_2, n_7, n_4, n_9]$, $n = n_0$.

Algorithm BFS($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, $s \in \mathcal{N}$):

```
marked := {n ↦ (n ≠ s) | n ∈ N}.
Q := a queue holding only s.
while ¬EMPTY(Q) do
    n := DEQUEUE(Q).
    for all (n, m) ∈ E do
        if ¬marked[m] then
            marked[m] := true.
            ENQUEUE(S, m).
```

$marked =$

n_0	true
n_1	false
n_2	true
n_3	true
n_4	true
n_5	false
n_6	false
n_7	true
n_8	false
n_9	true



Traversing undirected graphs: Breadth-first

$Q : [n_7, n_4, n_9]$, $n = n_2$.

Algorithm BFS($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, $s \in \mathcal{N}$):

$marked := \{n \mapsto (n \neq s) \mid n \in \mathcal{N}\}$.
Q := a queue holding only s.
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      Enqueue(S, m).

$marked =$

n_0	true
n_1	false
n_2	true
n_3	true
n_4	true
n_5	false
n_6	false
n_7	true
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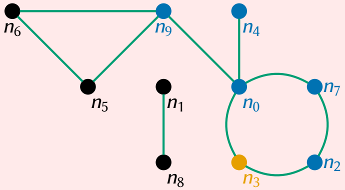
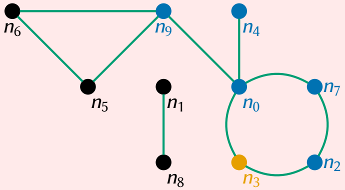


Traversing undirected graphs: Breadth-first

$Q : [n_4, n_9]$, $n = n_7$.

Algorithm BFS($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, $s \in \mathcal{N}$):

```
marked := {n ↦ (n ≠ s) | n ∈ N}.
Q := a queue holding only s.
while ¬EMPTY(Q) do
    n := DEQUEUE(Q).
    for all (n, m) ∈ E do
        if ¬marked[m] then
            marked[m] := true.
            ENQUEUE(S, m).
```



marked =

n_0	true
n_1	false
n_2	true
n_3	true
n_4	true
n_5	false
n_6	false
n_7	true
n_8	false
n_9	true

Traversing undirected graphs: Breadth-first

$Q : [n_9]$, $n = n_4$.

Algorithm BFS($\mathcal{G} = (\mathcal{N}, \mathcal{E}), s \in \mathcal{N}$):

$marked := \{n \mapsto (n \neq s) \mid n \in \mathcal{N}\}$.
$Q :=$ a queue holding only s.
while $\neg$EMPTY(Q) **do**
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    for all $(n, m) \in \mathcal{E}$ **do**
        if $\neg marked[m]$ **then**
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n_1	false
n_2	true
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n_5	false
n_6	false
n_7	true
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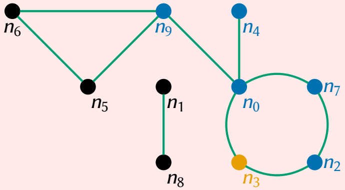

Traversing undirected graphs: Breadth-first

$Q : [n_6, n_5], n = n_9.$

Algorithm BFS($\mathcal{G} = (\mathcal{N}, \mathcal{E}), s \in \mathcal{N}$):

$marked := \{n \mapsto (n \neq s) \mid n \in \mathcal{N}\}$.
$Q :=$ a queue holding only s.
while $\neg$EMPTY(Q) **do**
  $n :=$ DEQUEUE(Q).
  for all $(n, m) \in \mathcal{E}$ **do**
    if $\neg marked[m]$ **then**
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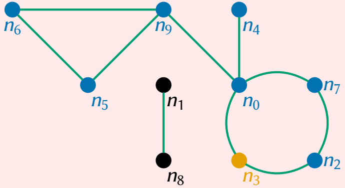

$marked =$

n_0	true
n_1	false
n_2	true
n_3	true
n_4	true
n_5	true
n_6	true
n_7	true
n_8	false
n_9	true

Traversing undirected graphs: Breadth-first

$Q : [n_5]$, $n = n_6$.

Algorithm BFS($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, $s \in \mathcal{N}$):

```
marked := {n ↦ (n ≠ s) | n ∈ 𝒩}.
Q := a queue holding only s.
while ¬Empty(Q) do
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    for all (n, m) ∈ ℰ do
        if ¬marked[m] then
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```

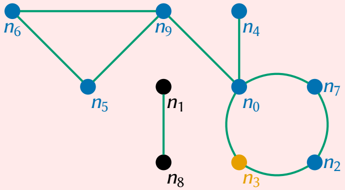

marked =

n_0	true
n_1	false
n_2	true
n_3	true
n_4	true
n_5	true
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Traversing undirected graphs: Breadth-first

$Q : [], n = n_5.$

Algorithm BFS($\mathcal{G} = (\mathcal{N}, \mathcal{E}), s \in \mathcal{N}$):

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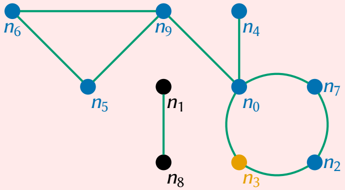

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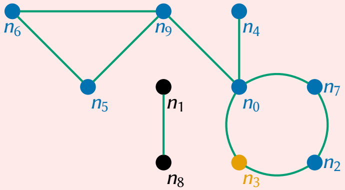
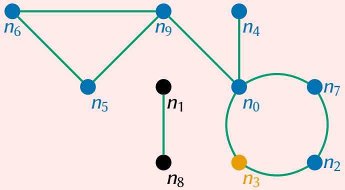
Traversing undirected graphs: Breadth-first



What can we learn from this breadth-first search?

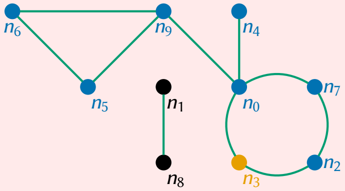
Traversing undirected graphs: Breadth-first



What can we learn from this breadth-first search?

- We found all nodes to which n_3 is *connected* (nodes one can reach from n_3).

Traversing undirected graphs: Breadth-first



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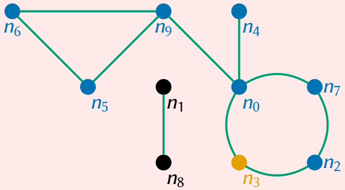
Traversing undirected graphs: Breadth-first



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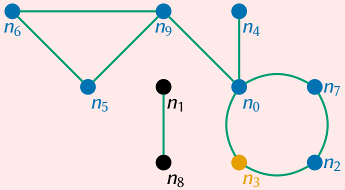
Breadth-first search is *similar* to depth-first search!

Traversing undirected graphs: Breadth-first



Complexity

- We need $|\mathcal{N}|$ memory for *marked*.
- We inspect each node once and traverse each edge once: $\Theta(|\mathcal{N}| + |\mathcal{E}|)$ (if we use the adjacency list representation).

Problem: Single-source shortest path

Problem

Given an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ *without weight* and node $s \in \mathcal{N}$, find a shortest path from node s to all nodes s can reach.

Problem: Single-source shortest path

Problem

Given an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ *without weight* and node $s \in \mathcal{N}$, find a shortest path from node s to all nodes s can reach.

Observe

Breadth-first search visits nodes on increasing distance to s.
First: all nodes at distance 1, then all nodes at distance 2, ...

Problem: Single-source shortest path

Problem

Given an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ *without weight* and node $s \in \mathcal{N}$, find a shortest path from node s to all nodes s can reach.

Algorithm BFS-SSSP($\mathcal{G}$, $s \in \mathcal{N}$):

$\mathit{distance} := \{n \mapsto \infty \mid n \in \mathcal{N}\}$.
$\mathit{distance}[s] := 0$.
Q := a queue holding only s.
while $\neg\textsc{Empty}(Q)$ **do**
    $n := \textsc{Dequeue}(Q)$.
    for all $(n, m) \in \mathcal{E}$ **do**
        if $\mathit{distance}[m] = \infty$ **then**
            $\mathit{distance}[m] := \mathit{distance}[n] + 1$.
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Problem: Single-source shortest path

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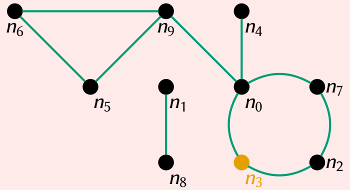

Problem: Single-source shortest path

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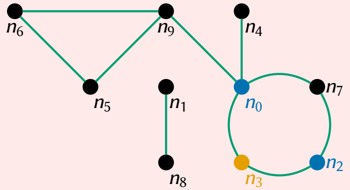

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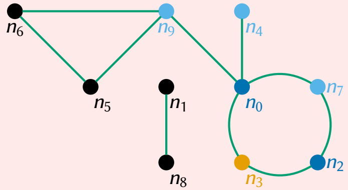

Problem: Single-source shortest path

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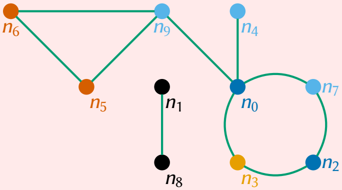

Problem: Single-source shortest path

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$distance[s] := 0$.

Q := a queue holding only s.

while $\neg$Empty(Q) **do**

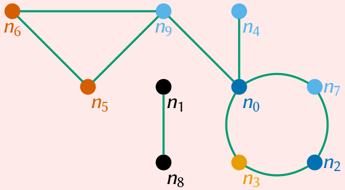
  $n :=$ Dequeue(Q).

  for all $(n, m) \in \mathcal{E}$ **do**

    if $distance[m] = \infty$ **then**

      $distance[m] := distance[n] + 1$.   *include (n, m) in the shortest path.*

      Enqueue(Q, m).



Problem: Single-source shortest path

Algorithm BFS-SSSP($\mathcal{G}$, $s \in \mathcal{N}$):

$distance := \{n \mapsto \infty \mid n \in \mathcal{N}\}$.
$distance[s] := 0$.
$Q :=$ a queue holding only s.
while $\neg$Empty(Q) **do**
    $n :=$ Dequeue(Q).
    for all $(n, m) \in \mathcal{E}$ **do**
        if $distance[m] = \infty$ **then**
            $distance[m] := distance[n] + 1$.   *include (n, m) in the shortest path.*
            Enqueue(Q, m).

We inspect each node once and traverse each edge once: $\Theta(|\mathcal{N}| + |\mathcal{E}|)$.

Traversing directed graphs: Depth-first

Algorithm DFS-R($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, *marked*, $n \in \mathcal{N}$):

for all $(n, m) \in \mathcal{E}$ **do**
    if $\neg marked[m]$ **then**
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        DFS-R($\mathcal{G}$, *marked*, m).

Algorithm DepthFirstR($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, $s \in \mathcal{N}$):

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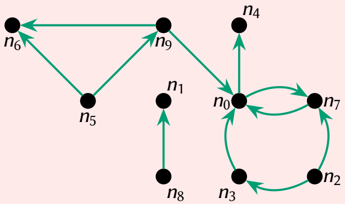

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Same algorithm as for *undirected* graphs.

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```
for all (n, m) ∈ ℰ do
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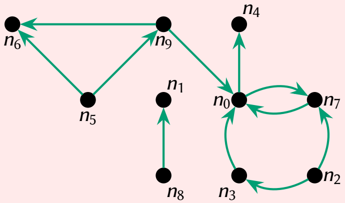

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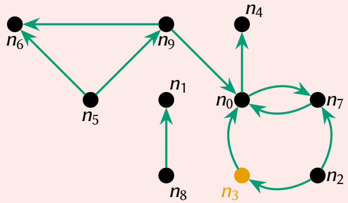

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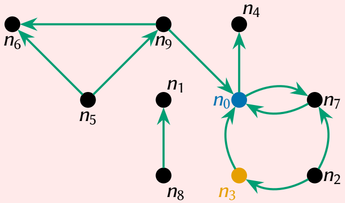

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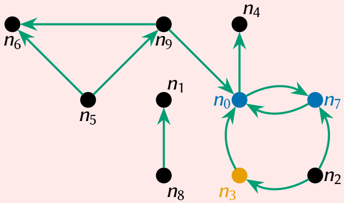

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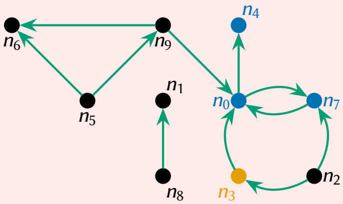

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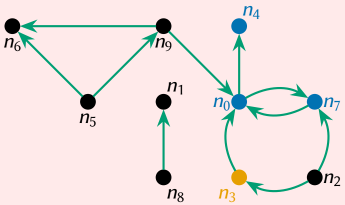
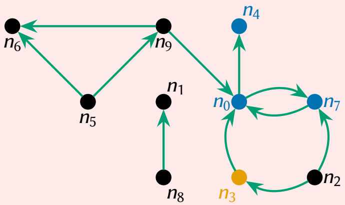


What can we learn from this depth-first search?

Traversing directed graphs: Depth-first



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- We found all nodes to which n_3 is *strongly connected* (nodes one can reach from n_3).

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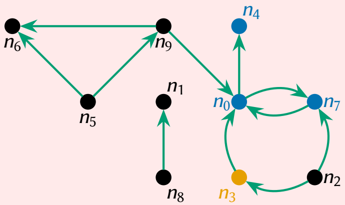

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- We found all nodes to which n_3 is *strongly connected* (nodes one can reach from n_3).
- The order of recursive calls was:

$$n = n_3, n_0, \begin{cases} n_7; \\ n_4. \end{cases}$$

This order provides a path from n_3 to *every* node it is strongly connected to!

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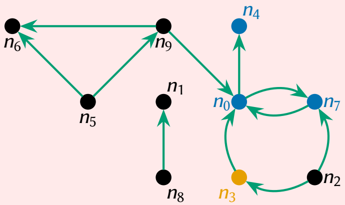

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  This order provides a path from n_3 to *every* node it is strongly connected to!
- Depth-first search does *not* tell us whether a graph is strongly connected!

Traversing directed graphs: Breadth-first

Algorithm BFS($\mathcal{G} = (\mathcal{N}, \mathcal{E}), s \in \mathcal{N}$):

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marked := {n ↦ (n ≠ s) | n ∈ 𝒩}.
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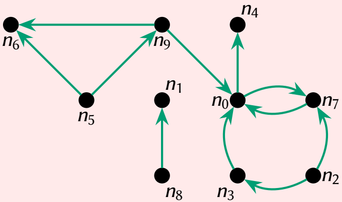

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Same algorithm as for *undirected* graphs.

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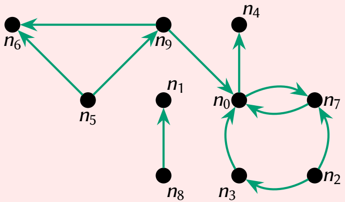

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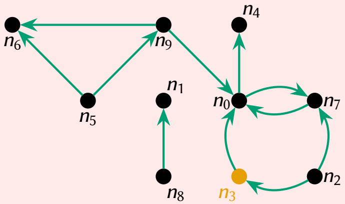

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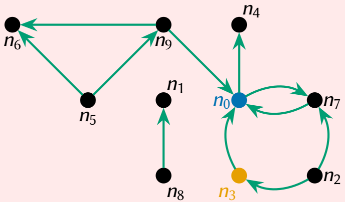

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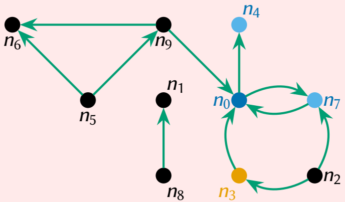

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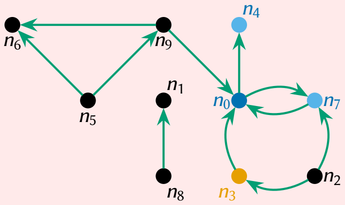
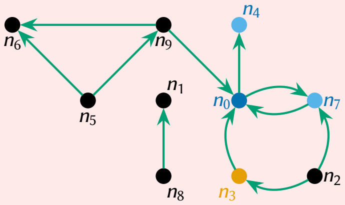


What can we learn from this breadth-first search?

Traversing directed graphs: Breadth-first



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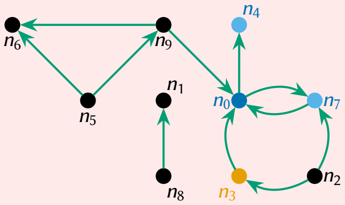
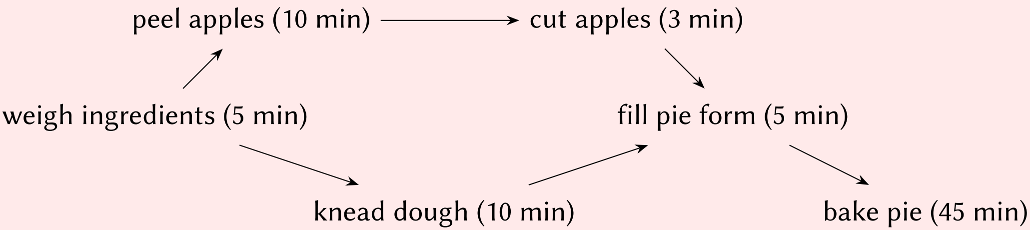


What can we learn from this breadth-first search?

- We found all nodes to which n_3 is *strongly connected* (nodes one can reach from n_3).
- We can also easily find shortest directed paths node n_3 can reach.

Problem: Cyclic dependencies



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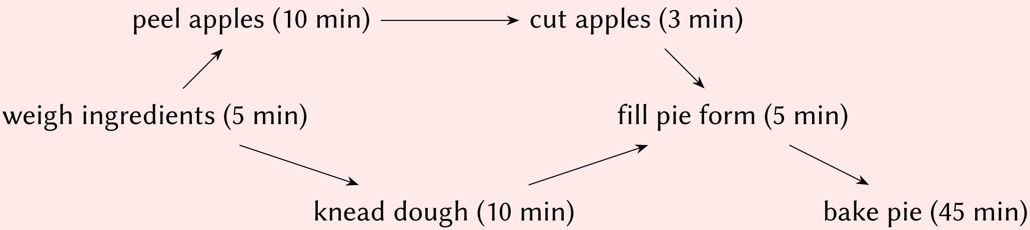

Problem

We can only satisfy the schedule if the graph of dependencies is *acyclic*:
Acylcic graph: there are *no* directed cycles.

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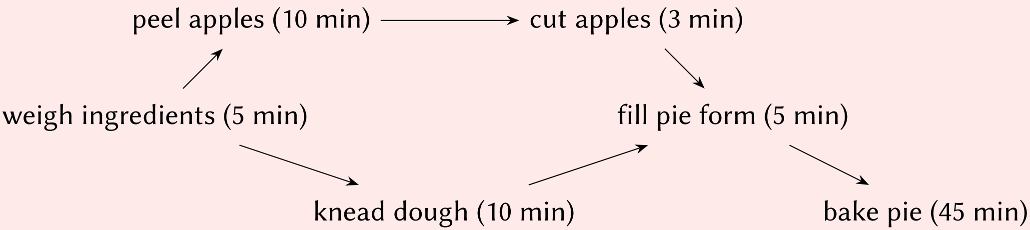

Problem

We can only satisfy the schedule if the graph of dependencies is *acyclic*:
Acylcic graph: there are *no* directed cycles.

There is no path with at-least one edge from a node n to itself.

Problem: Cyclic dependencies

Find a *directed* cycle: a path from a node to itself

Consider a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.

Problem: Cyclic dependencies

Find a *directed* cycle: a path from a node to itself

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Assume node m is the fist node visited during depth-first search with a path to itself.

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- From m, we will visit a path $mn'_1 \dots n'_j w$ to some node w such that node w has an edge to node n. *Why?*

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  - We could have started at a node $s \neq m$, however. But: the nodes $sn_1 \dots n_i$ are not part of a cycle. Hence, m cannot reach them!

Problem: Cyclic dependencies

Find a *directed* cycle: a path from a node to itself

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  - We could have started at a node $s \neq m$, however.
    But: the nodes $sn_1 \dots n_i$ are not part of a cycle. Hence, m cannot reach them!

Conclusion. Depth-first search can find cycles:
We simply have to detect nodes that reach themselves!

Problem: Cyclic dependencies

Find a *directed* cycle: a path from a node to itself

Consider a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.

Algorithm DFS-C-R($\mathcal{G}$, *marked*, $n \in \mathcal{N}$):

```
for all (n, m) ∈ E do
    if marked[m] = unmarked then
        marked[m] := inspecting.
        DFS-C-R(G, marked, m).
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Algorithm DFS-Cycle($\mathcal{G}$):

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marked := {n ↦ unmarked | n ∈ N}.
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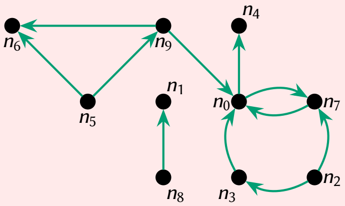

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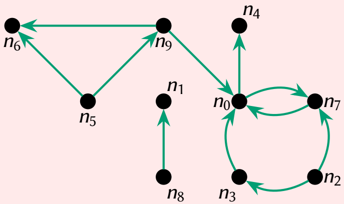

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Called with $n = n_3$.

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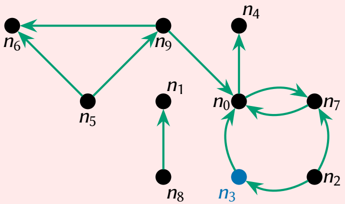

Problem: Cyclic dependencies

Find a *directed* cycle: a path from a node to itself

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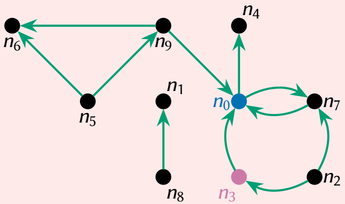

Problem: Cyclic dependencies

Find a *directed* cycle: a path from a node to itself

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Called with $n = n_3, n_0, n_4$.

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for all n ∈ N do
    if marked[n] = unmarked then
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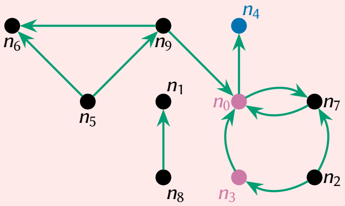
        DFS-C-R(G, marked, n).
```



Problem: Cyclic dependencies

Find a *directed* cycle: a path from a node to itself

Consider a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.

Called with $n = n_3, n_0, n_7$.

Algorithm DFS-C-R($\mathcal{G}$, *marked*, $n \in \mathcal{N}$):

for all $(n, m) \in \mathcal{E}$ **do**
  if $marked[m] = unmarked$ **then**
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    Found a path that contains a cycle.
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Algorithm DFS-Cycle($\mathcal{G}$):

$marked := \{n \mapsto unmarked \mid n \in \mathcal{N}\}$.
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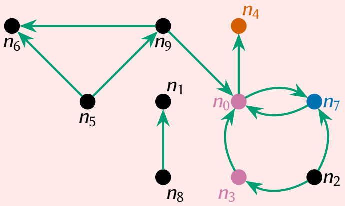

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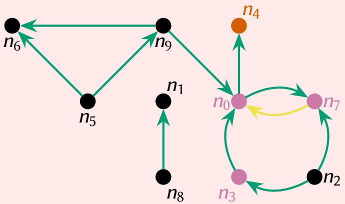
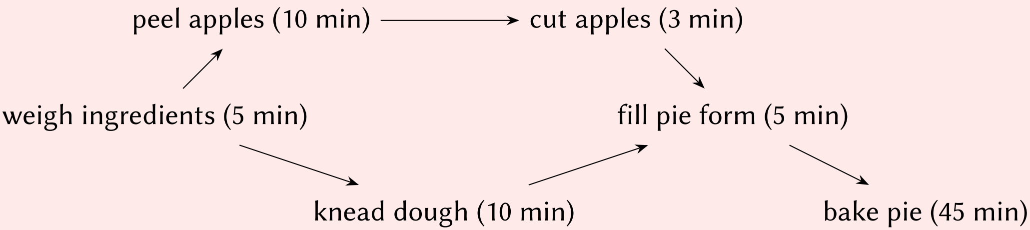


Problem: Ordering tasks



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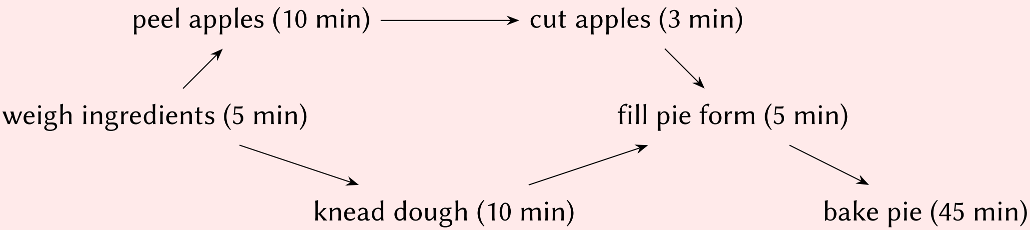

Problem

To schedule the tasks, we need to order tasks based on their dependencies.

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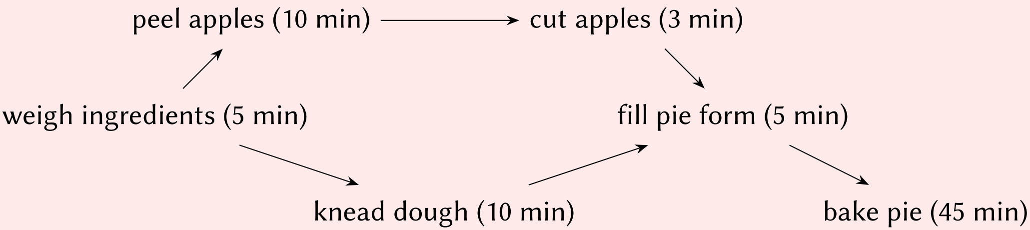

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Topological order: an order on nodes such that,
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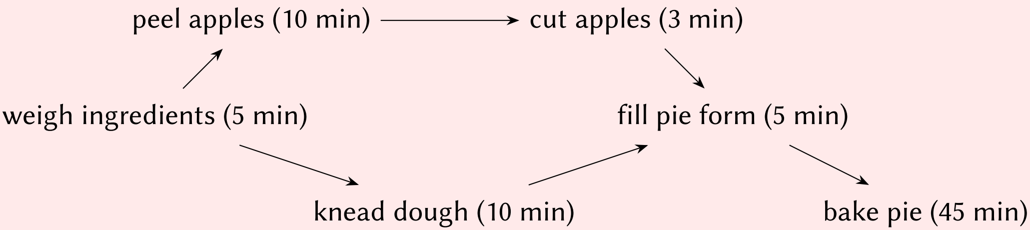

Problem

To schedule the tasks, we need to order tasks based on their dependencies.

Topological order: an order on nodes such that,
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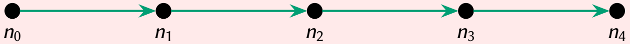
We *cannot* have a topological order if the graph is cyclic.

Problem: Ordering tasks

Determine a *topological order*

Depth-first search seems related: if we reach node n after inspecting m,
then m should definitely come before n in the order.

Consider first starting depth-first search at n_2, and then starting at n_0.



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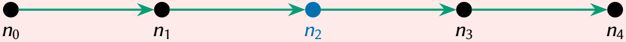

We inspect the nodes in the order: n_2.

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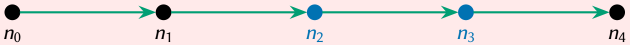

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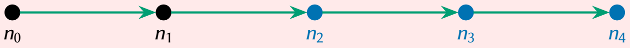

We inspect the nodes in the order: n_2, n_3, n_4.

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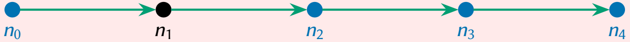

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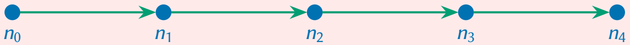

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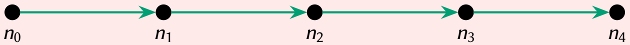

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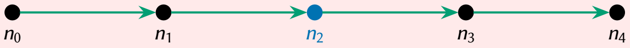

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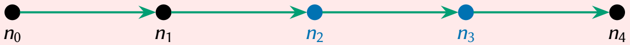

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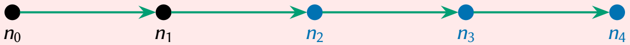

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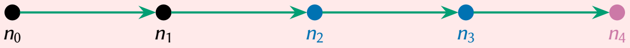

We inspect the nodes in the order: n_2, n_3, n_4, n_0, n_1.

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We finish inspecting the nodes in the order n_4.

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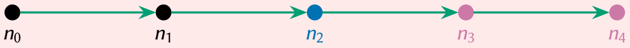

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We finish inspecting the nodes in the order n_4, n_3.

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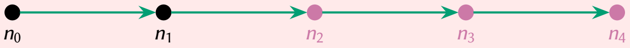

We inspect the nodes in the order: n_2, n_3, n_4, n_0, n_1.

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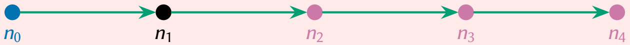

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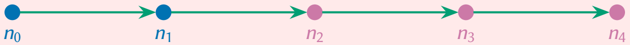

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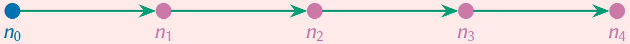

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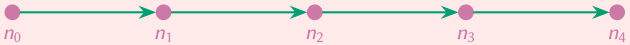

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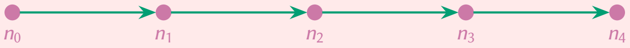

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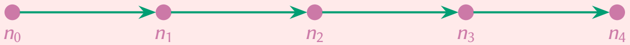

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We need to prove that this is correct!

Problem: Ordering tasks

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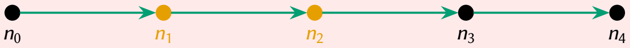


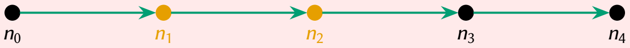
We annotate depth-first search to collect ordering information:
When we *finish* inspecting a node, we add it to the *front* of our order.

Theorem

Let $(m, n) \in \mathcal{E}$ be an edge in an acyclic graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.
Any depth-first search on $\mathcal{G}$ will finish inspecting n before m (hence, m is placed before n in our order).

Problem: Ordering tasks

Determine a *topological order*



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Proof. We consider two cases:

- *When we run depth-first search for m, n is already marked.*
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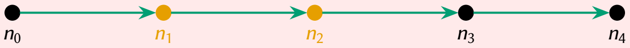
Problem: Ordering tasks

Determine a *topological order*



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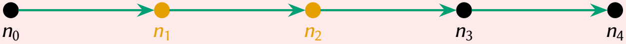

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  Graph is acylcic: n cannot reach m, hence we finished inspecting n already.
- *When we run depth-first search for m, n is not yet marked*.
  We find n while inspecting m, hence we finished inspecting n before m.

Problem: Ordering tasks

Algorithm DFS-TS-R($\mathcal{G} = (\mathcal{N}, \mathcal{E})$, *marked*, $n \in \mathcal{N}$, *order*):

for all $(n, m) \in \mathcal{E}$ **do**
    if $\neg$*marked*[m] **then**
        marked[m] := `true`.
        DFS-TS-R($\mathcal{G}$, *marked*, m, *order*).
Add n to the front of *order*.

Algorithm TopologicalSort($\mathcal{G} = (\mathcal{N}, \mathcal{E})$):

marked, *order* := $\{n \mapsto$ `false` $\mid n \in \mathcal{N}\}$, [].
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        DFS-TS-R($\mathcal{G}$, *marked*, n, *order*).
return *order*.

Problem: Ordering tasks

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We can easily integrate a cycle-detection step into TopologicalSort.

Problem: Reverse reachability

Consider a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ and node s.
Depth-first search can find all nodes reachable from node s.

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How to find all nodes that can reach node s?
E.g., in a one-way communication network:
which participants can communicate messages to s?

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Solution

Reverse all edges in $\mathcal{G}$ and perform depth-first search on the resulting graph.
Hence, $\textsc{DepthFirstR}(\mathcal{G}', s)$ with $\mathcal{G}' = (\mathcal{N}, \{(n, m) \mid (m, n) \in \mathcal{E}\})$.

Problem: Healthy network

Problem

Consider a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ in which

- the nodes $\mathcal{N}$ represent network devices; and
- the edges $\mathcal{E}$ are network connections.

Can all network devices communicate with all other network devices?

Problem: Healthy network

Problem

Consider a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ in which

- the nodes $\mathcal{N}$ represent network devices; and
- the edges $\mathcal{E}$ are network connections.

Can all network devices communicate with all other network devices?

Problem

Consider a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.
A graph is *strongly connected* if all node pairs are strongly connected.

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Now consider an arbitrary node $s \in \mathcal{N}$.

1. All nodes must have a path to node s.
2. Node s must have a path to all nodes.

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We can route all paths proving "strongly connected" via node s.

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There must be a directed path between all pairs of nodes.

Now consider an arbitrary node $s \in \mathcal{N}$.

1. All nodes must have a path to node s. $\longrightarrow$ Use *reverse reachability*.
2. Node s must have a path to all nodes. $\longrightarrow$ Use *reachability*.

We can route all paths proving "strongly connected" via node s.

Solution

Use *reverse reachability* and *reachability*.
Both can be done via depth-first search.

Problem: Subcommunities

Problem

Consider a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ in which

- the nodes $\mathcal{N}$ represent social media accounts; and
- the edges $\mathcal{E}$ are interactions between accounts.

We want to find subcommunities (and echo chambers) by looking groups of accounts that all have direct-or-indirect interactions with each other.

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Problem

Consider a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.
Find all *strongly connected components*.

Problem: Subcommunities

Observations

For each $n \in \mathcal{N}$, let $\mathrm{scc}(n)$ be all nodes in the strongly connected component of n.

Consider the graph $\mathcal{G}_{\mathrm{SCC}} = (\mathcal{N}_{\mathrm{SCC}}, \mathcal{E}_{\mathrm{SCC}})$ obtained by *merging* the strongly connected components in $\mathcal{G}$:

- $\mathcal{N}_{\mathrm{SCC}} = \{\mathrm{scc}(n) \mid n \in \mathcal{N}\}$;
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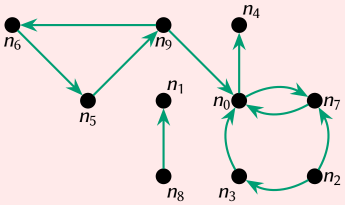

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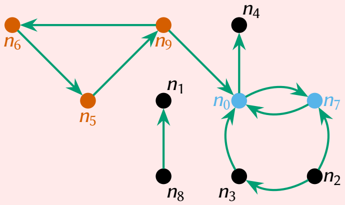

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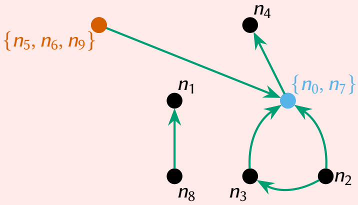

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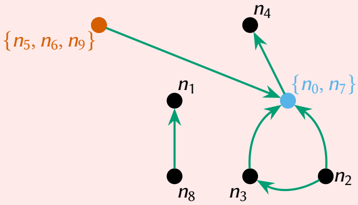

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A node order consistent with a topological sort of $\mathcal{G}_{\text{scc}}$:
the strongly connected components stick together.

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Question: What would a *topological sort* of $\mathcal{G}$ produce?
A node order consistent with a topological sort of $\mathcal{G}_{\text{scc}}$:
the strongly connected components stick together.

We just do not know where one strongly connected component ends and the next begins.

Problem: Subcommunities

Algorithm $\textsc{StronglyConnectedComponent}(\mathcal{G} = (\mathcal{N}, \mathcal{E}))$:

Let $n_0, \ldots, n_{|\mathcal{N}|}$ be a topological sort of $\mathcal{N}$.
$marked := \{n \mapsto \texttt{false} \mid n \in \mathcal{N}\}$.
for $i := 0$ upto $|\mathcal{N}|$ **do**
    if $\neg marked[n_i]$ **then**
        DFS-R$((\mathcal{N}, (\{(n, m) \mid (m, n) \in \mathcal{E}\})), marked, n_i)$ (*reverse reachability*).

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        n_i is the start of a strongly connected component.
        Find all nodes not-yet-visited that can *reach* node n_i.
        DFS-R$((\mathcal{N}, (\{(n, m) \mid (m, n) \in \mathcal{E}\})), marked, n_i)$ (*reverse reachability*).

Problem: Subcommunities

Algorithm StronglyConnectedComponent($\mathcal{G} = (\mathcal{N}, \mathcal{E})$):

```
Let n_0, ..., n_|N| be a topological sort of N.
marked := {n ↦ false | n ∈ N}.
for i := 0 upto |N| do
    if ¬marked[n_i] then
        n_i is the start of a strongly connected component.
        Find all nodes not-yet-visited that can reach node n_i.
        DFS-R((N, ({(n, m) | (m, n) ∈ E})), marked, n_i) (reverse reachability).
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A node that can reach n_i comes before n_i in the topological sort
unless it is part of the same strongly connected component!

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The book presents a variation of the above:
they perform a reverse-topological sort instead of performing reverse reachability.

Problem: Indirect flight connections

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Consider a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ in which

- the nodes $\mathcal{N}$ represent airports; and
- the edges $\mathcal{E}$ are flights between airports.

Construct the edge relation that relates airports m to n if one can fly from m to n (via zero-or-more stops):

$$\{(m, n) \mid \text{there is a sequence of flights connecting } m \text{ to } n\}.$$

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Definition

The *transitive closure* of a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is the graph $\mathcal{G}_{\text{tc}} = (\mathcal{N}, \mathcal{E}_{\text{tc}})$ with

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Complexity

- Runtime complexity is $\Theta\left(|\mathcal{N}|(|\mathcal{N}| + |\mathcal{E}|)\right)$: we run $|\mathcal{N}|$ depth-first searches.
- Memory complexity is $\Theta\left(|\mathcal{N}| + |\mathcal{E}_{\text{tc}}|\right)$: $|\mathcal{E}_{\text{tc}}|$ is likely to be $\Theta\left(|\mathcal{N}|^2\right)$.

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Can we do significantly better? Huge open research question!