Sorting

SFWRENG 2CO3: Data Structures and Algorithms

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Using MERGE-like algorithms

Consider the following variant of MERGE.

Algorithm MERGE(L_1, L_2):

Input: L_1 and L_2 are ordered lists of distinct values.

output := $\emptyset$.
$i_1, i_2 := 0, 0$.
while $i_1 < |L_1|$ **or** $i_2 < |L_2|$ **do**
  if $(i_1 < |L_1|$ **and** $i_2 < |L_2|)$ **and also** $L_1[i_1] = L_2[i_2]$ **then**
    Add $L_1[i_1]$ to *output*.
    $i_1, i_2 := i_1 + 1, i_2 + 1$.
  else if $i_2 = |L_2|$ **or else** $(i_1 < |L_1|$ **and also** $L_1[i_1] < L_2[i_2])$ **then**
    Add $L_1[i_1]$ to *output*.
    $i_1 := i_1 + 1$.
  else $L_1[i_1] > L_2[i_2]$
    Add $L_2[i_2]$ to *output*.
    $i_2 := i_2 + 1$.
return *output*. /* return $L_1 \cup L_2$. */

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  if ($i_1 < |L_1|$ **and** $i_2 < |L_2|$) **and also** $L_1[i_1] = L_2[i_2]$ **then**
    Add $L_1[i_1]$ to *output*.
    $i_1, i_2 := i_1 + 1, i_2 + 1$.
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    ~~Add $L_1[i_1]$ to *output*~~.
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  else $L_1[i_1] > L_2[i_2]$
    ~~Add $L_2[i_2]$ to *output*~~.
    $i_2 := i_2 + 1$.
return *output*. /* return $L_1 \cap L_2$. */

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return *output*. /* return $L_1 \setminus L_2$. */

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  else $L_1[i_1] > L_2[i_2]$
    Add $L_2[i_2]$ to *output*.
    $i_2 := i_2 + 1$.
return *output*. /* return $(L_1 \cup L_2) \setminus (L_1 \cap L_2)$. */

Using Merge-like algorithms

Consider relations enrolled(c, *student*) and teaches(c, *faculty*), *ordered* on course *course*.

Problem

Compute all pairs (*student*, *faculty*) such that *faculty* is a teacher of *student*.

Solutions

- A nested-loop join: $\Theta(|\text{enrolled}| \cdot |\text{teaches}|)$.
- Using binary search: $\Theta(|\text{enrolled}| \cdot \log_2(|\text{teaches}|) + |\mathit{result}|)$.

Can we do better?

Using MERGE-like algorithms

Consider relations enrolled(c, *student*) and teaches(c, *faculty*), *ordered* on course *course*.

Algorithm ETMERGEJOIN(*enrolled*, *teaches*):

output $:= \emptyset$.
$i_1, i_2 := 0, 0$.
while $i_1 < |enrolled|$ **and** $i_2 < |teaches|$ **do**
    if enrolled$[i_1].c$ = teaches$[i_2].c$ **then**
        A potential join output!
        Need to find all enrolled students for course enrolled$[i_1].c$.
        Need to find all teaching faculty for course teaches$[i_2].c$.

    else if enrolled$[i_1].c$ < teaches$[i_2].c$ **then**
        $i_1 := i_1 + 1$.
    else enrolled$[i_1].c$ < teaches$[i_2].c$
        $i_2 := i_2 + 1$.
return *output*. /* return pairs (s, f) such that f is a teacher of s. */

Using MERGE-like algorithms

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output := $\emptyset$.
$i_1, i_2 := 0, 0$.
while $i_1 < |enrolled|$ **and** $i_2 < |teaches|$ **do**
    if enrolled[i_1].c = teaches[i_2].c **then**
        j_1 := first j with either j = |enrolled| or else enrolled[j].$c \neq$ enrolled[i_1].c.
        j_2 := first j with either j = |teaches| or else teaches[j].$c \neq$ teaches[i_2].c.
        Add all (s, f) with $(c_1, s) \in$ enrolled[i_1, j_1) and $(c_2, f) \in$ teaches[i_2, j_2) to *output*.
        $i_1, i_2 := j_1, j_2$.
    else if enrolled[i_1].c < teaches[i_2].c **then**
        $i_1 := i_1 + 1$.
    else enrolled[i_1].c < teaches[i_2].c
        $i_2 := i_2 + 1$.
return *output*. /* return pairs (s, f) such that f is a teacher of s. */

Using Merge-like algorithms

Consider relations enrolled(c, *student*) and teaches(c, *faculty*), *ordered* on course *course*.

Algorithm ETMergeJoin(*enrolled*, *teaches*):

```
output := ∅.
i1, i2 := 0, 0.
while i1 < |enrolled| and i2 < |teaches| do
    if enrolled[i1].c = teaches[i2].c then
        j1 := first j with either j = |enrolled| or else enrolled[j].c ≠ enrolled[i1].c.
        j2 := first j with either j = |teaches| or else teaches[j].c ≠ teaches[i2].c.
        Add all (s, f) with (c1, s) ∈ enrolled[i1, j1) and (c2, f) ∈ teaches[i2, j2) to output.
        i1, i2 := j1, j2.
```

Complexity

- The *merge*-part visits every value in enrolled and teaches once.
- The *join*-part only visits those pairs of values necessary for the result.

Hence, the complexity is $\Theta(|enrolled| + |teaches| + |result|)$.

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- Using binary search: $\Theta(|\text{enrolled}| \cdot \log_2(|\text{teaches}|) + |\mathit{result}|)$.
- Using merge join: $\Theta(|\mathit{enrolled}| + |\mathit{teaches}| + |\mathit{result}|)$.

Stable sorting

Consider a list *enrolled* of enrollment data with schema

$$\text{enrolled}(\mathit{dept}, \mathit{code}, \mathit{sid}, \mathit{date}).$$

If we add enrollment data to *the end of the list*, then enrolled is always sorted on *date*.

Problem

Group enrolled on (dept, code) and within each group sort enrollments on *date*.

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Group enrolled on ($\mathit{dept}, \mathit{code}$) and within each group sort enrollments on *date*.

Brute-force solution: Lexicographical sorting on ($\mathit{dept}, \mathit{code}, \mathit{date}$)

Let $(d_1, c_1, s_1, t_1), (d_2, c_2, s_2, t_2) \in \text{enrolled}$. We use the comparison

$$(d_1, c_1, s_1, t_1) \mathrel{\textit{before}} (d_2, c_2, s_2, t_2) \text{ if } (d_1 < d_2) \lor ((d_1 = d_2) \land (c_1 < c_2)) \lor \\ ((d_1 = d_2) \land (c_1 = c_2) \land (t_1 < t_2)).$$

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$$((d_1 = d_2) \land (c_1 = c_2) \land (t_1 < t_2)).$$

Downside: During sorting, we end up throwing away the existing ordering on *date*, and then we rebuild that order from scratch!

Stable sorting

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If we add enrollment data to *the end of the list*, then enrolled is always sorted on *date*.

Problem

Group enrolled on (dept, code) and within each group sort enrollments on *date*.

Better solution: Use a *stable sort algorithm*

A *stable sort algorithm* maintains the relative order of "equal values".

Let $(d_1, c_1, s_1, t_1), (d_2, c_2, s_2, t_2) \in$ enrolled. If we sort enrolled using a *stable sort algorithm* using the comparison

$$(d_1, c_1, s_1, t_1) \textit{ before } (d_2, c_2, s_2, t_2) \text{ if } (d_1 < d_2) \lor ((d_1 = d_2) \land (c_1 < c_2))$$

then within each (dept, code)-group, enrollments remain ordered on *date* for free!

Stable sorting

Definition

Let L be a list that is already ordered with respect to some attributes $a_1, \ldots, a_n$.
Consider a sort step S that re-orders L based on other attributes $b_1, \ldots, b_m$.

We say that the sort step S is *stable* if, for every value $r_1 \in L$ and $r_2 \in L$ such that r_1 originally came before r_2 and r_1 and r_2 agreee on attributes $b_1, \ldots, b_m$, the resulting re-ordered list will still have r_1 come before r_2.

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Question: Have we already seen stable sort algorithms?

Yes: SelectionSort, InsertionSort, and MergeSort.

Note: even minor changes to these algorithms will make them non-stable!
(e.g., changing $<$ into $\leq$).

Intermezzo: Recurrence trees

In a recurrence tree

- nodes labeled N represent a *function call* with “input size N”;
- the children of a node represent *recursive calls*;
- per node, we can determine *the work* within that call (besides recursion);
- per depth, we can determine the *total work for that depth*;
- by *summing over all depths*: the total complexity.

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- by *summing over all depths*: the total complexity.

We already saw two examples: LowerBoundRec and MergeSortR.

Intermezzo: Recurrence trees

Example: the *Fibonacci numbers*

$$fib(N) = \begin{cases} 1 & \text{if } N = 1 \text{ or } N = 2; \\ fib(N-1) + fib(N-2) & \text{if } N > 2. \end{cases}$$

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Prove that $fib(N) \leq 2^N$

Simplication: $fib(i-2) \leq fib(i-1)$.

	Number	Cost	Total
N	$1 = 2^0$	1	$1 \cdot 1 = 1$

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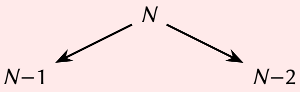

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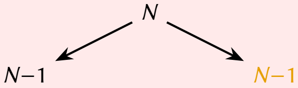

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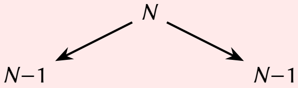

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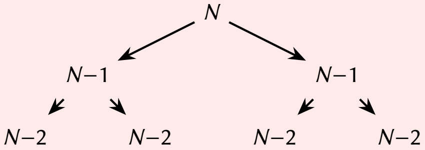

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$1 = 2^0$	1	$1 \cdot 1 = 1$
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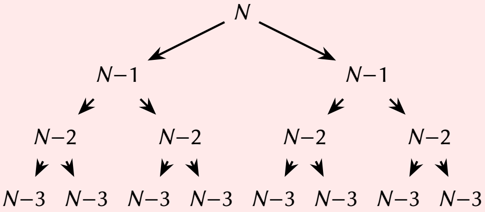

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$8 = 2^3$	1	$8 \cdot 1 = 8$

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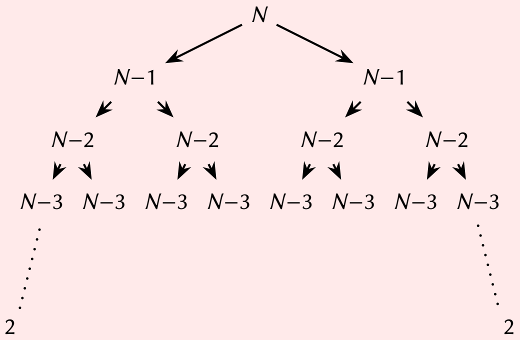

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$8 = 2^3$	1	$8 \cdot 1 = 8$
$\vdots$	$\vdots$	$\vdots$
2^i	1	$2^i \cdot 1 = 2^i$
$\vdots$	$\vdots$	$\vdots$

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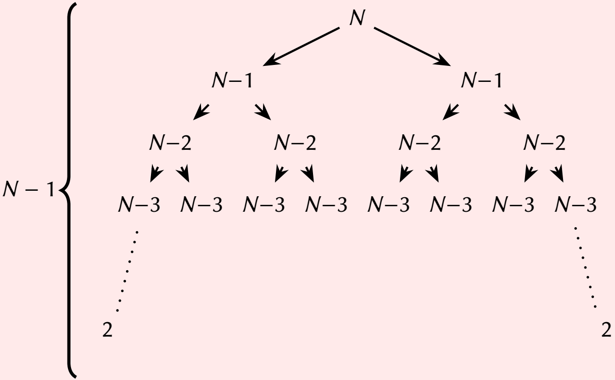

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2^i	1	$2^i \cdot 1 = 2^i$
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Intermezzo: Recurrence trees

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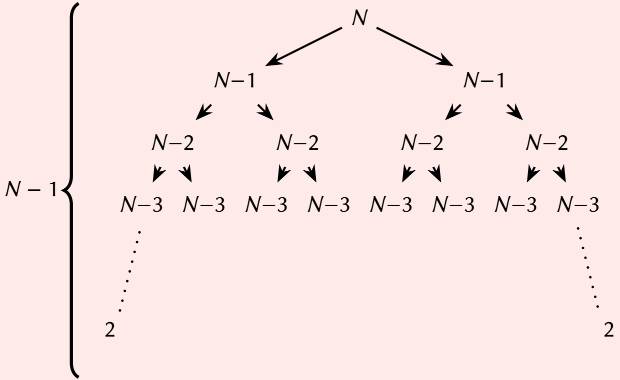

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$\vdots$	$\vdots$	$\vdots$
2^i	1	$2^i \cdot 1 = 2^i$
$\vdots$	$\vdots$	$\vdots$
		+
		$\sum_{i=0}^{N-2} 2^i$

Intermezzo: Recurrence trees

$$fib(N) = \begin{cases} 1 & \text{if } N = 1 \text{ or } N = 2; \\ fib(N-1) + fib(N-2) & \text{if } N > 2. \end{cases}$$

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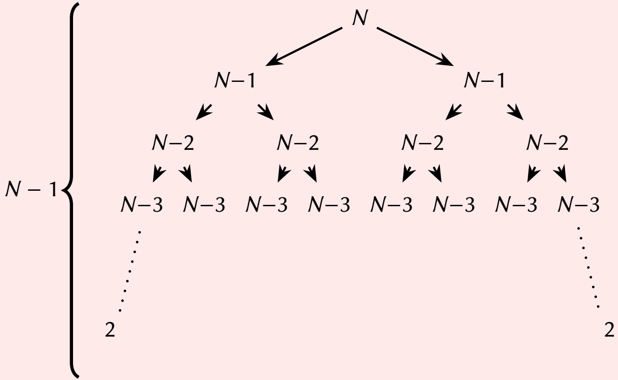

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$\vdots$	$\vdots$	$\vdots$
2^i	1	$2^i \cdot 1 = 2^i$
$\vdots$	$\vdots$	$\vdots$
		+

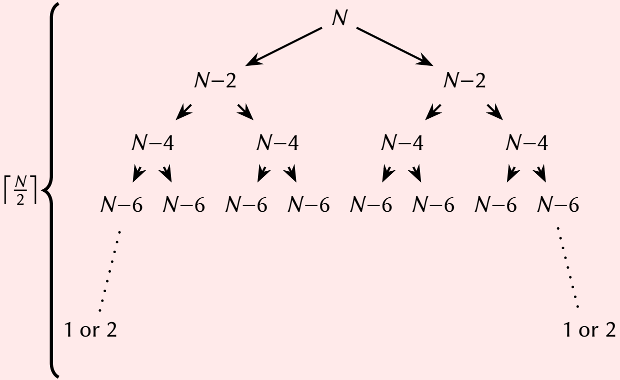
$$\sum_{i=0}^{N-2} 2^i = 2^{N-1} - 1$$

Intermezzo: Recurrence trees

$$fib(N) = \begin{cases} 1 & \text{if } N = 1 \text{ or } N = 2; \\ fib(N-1) + fib(N-2) & \text{if } N > 2. \end{cases}$$

Prove that $2^{\lceil \frac{N}{2} \rceil} \leq fib(N)$

Simplication: $fib(i-1) \geq fib(i-2)$.



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		+
		$\sum_{i=0}^{\lceil \frac{N}{2} \rceil} 2^i = 2^{\lceil \frac{N}{2} \rceil + 1} - 1$

Intermezzo: Recurrence trees

Example: the *Fibonacci numbers*

$$fib(N) = \begin{cases} 1 & \text{if } N = 1 \text{ or } N = 2; \\ fib(N-1) + fib(N-2) & \text{if } N > 2. \end{cases}$$

Via recurrence trees, we have proven that:

$$2^{\lceil \frac{N}{2} \rceil} \leq fib(N) \leq 2^N.$$

Intermezzo: The Master Theorem

Let $T(N)$ be a *recurrence* of the form

$$T(N) = \begin{cases} \textit{constant} & \text{if } \textit{base case}; \\ aT\left(\frac{N}{b}\right) + f(N) & \text{if } \textit{recursive case}, \end{cases}$$

with $a \geq 1$, $b > 1$, and we can read $\frac{N}{b}$ also as $\left\lceil \frac{N}{b} \right\rceil$ or $\left\lfloor \frac{N}{b} \right\rfloor$.

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Someone else has already proved this—so we can reuse the result!

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Example: Runtime complexity of LowerBoundRec

$$T(N) = \begin{cases} 4 & \text{if } N = 1; \\ T\left(\frac{N}{2}\right) + 8 & \text{if } N > 1. \end{cases}$$

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Case 2 yields: $T(N) = \Theta(N^{\log_2(1)} \log^1(N)) = \log(N)$.

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Example: Runtime complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N = 1; \\ T\left(\left\lfloor \frac{N}{2} \right\rfloor\right) + T\left(\left\lceil \frac{N}{2} \right\rceil\right) + N & \text{if } N > 1. \end{cases}$$

Intermezzo: The Master Theorem

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Example: Runtime complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N = 1; \\ T\left(\lfloor \frac{N}{2} \rfloor\right) + T\left(\lceil \frac{N}{2} \rceil\right) + N & \text{if } N > 1. \end{cases}$$

We have $a = 2$, $b = 2$, $f(N) = N = \Theta(N) = N^{\log_2(2)}$.

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Example: Runtime complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N = 1; \\ T\left(\lfloor \frac{N}{2} \rfloor\right) + T\left(\lceil \frac{N}{2} \rceil\right) + N & \text{if } N > 1. \end{cases}$$

We have $a = 2$, $b = 2$, $f(N) = N = \Theta(N) = N^{\log_2(2)}$.

Case 2 yields: $T(N) = \Theta(N^{\log_2(2)} \log^1(N)) = \Theta(N \log(N))$.

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A third example

$$T(N) = \begin{cases} 1 & \text{if } N = 1; \\ 7T\left(\left\lfloor \frac{N}{4} \right\rfloor\right) + N & \text{if } N > 1. \end{cases}$$

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Case 1 yields: $T(N) = \Theta(N^{\log_4(7)}) \approx \Theta(N^{1.40367\ldots})$.

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$$T(N) = \begin{cases} 1 & \text{if } N = 1; \\ 2T\left(\left\lfloor \frac{N}{2} \right\rfloor\right) + N^3 & \text{if } N > 1. \end{cases}$$

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Case 3 yields: $T(N) = \Theta(N^3)$.

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Feel free to use the Master Theorem, we will provide a copy during the final exam.

Can we do better than MergeSort?

Can we do better than MergeSort?

Algorithm CountSort($L[0 \ldots N]$):

Input: Each value in L is either 0 or 1.

$count_0 := 0$

for all $v \in L$ **do** Count number of 0's

    if $v = 0$ **then**

        $count_0 := count_0 + 1$.

for $i := 0$ to $count_0 - 1$ **do** Write the counted number of 0's

    $L[i] := 0$.

for $i := count_0$ to $N - 1$ **do** Write the remaining 1's

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Complexity: Linear ($\Theta(N)$ comparisons, $\Theta(N)$ changes)

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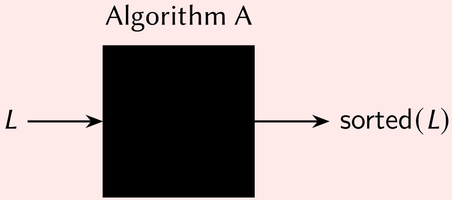
Complexity: Linear ($\Theta(N)$ comparisons, $\Theta(N)$ changes)

CountSort does *not* solve general-purpose sorting!

A lower bound for general-purpose sorting

Assume: We have a list $L[0 \ldots N)$ of N distinct values



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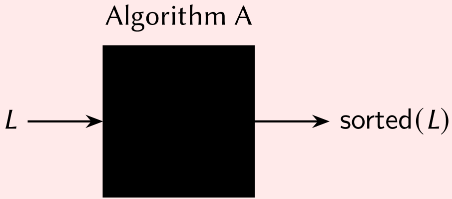

When is Algorithm A *general-purpose*?

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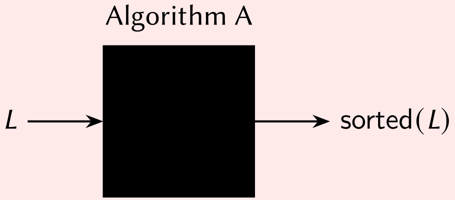

When is Algorithm A *general-purpose*?

- A uses *comparisons* to determine sorted order;
- A does *not require assumptions* on the value distribution in L.

A lower bound for general-purpose sorting

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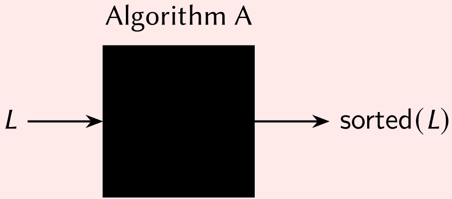

What do we know about *general-purpose* Algorithm A?

Consider lists $L_1 = [1, 3, 2, 4]$ and $L_2 = [1, 2, 3, 4]$.

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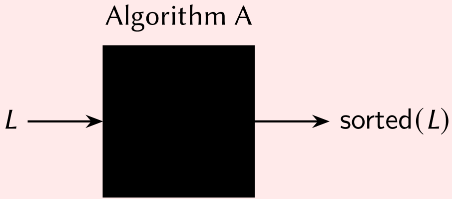

What do we know about *general-purpose* Algorithm A?

Consider lists $L_1 = [1, 3, 2, 4]$ and $L_2 = [1, 2, 3, 4]$.

- Algorithm A must perform *different* operations to order L_1 and L_2.

A lower bound for general-purpose sorting

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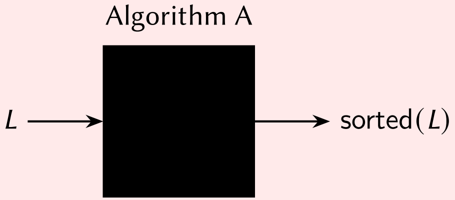

What do we know about *general-purpose* Algorithm A?

Consider lists $L_1 = [1, 3, 2, 4]$ and $L_2 = [1, 2, 3, 4]$.

- Algorithm A must perform *different* operations to order L_1 and L_2.
- Algorithm A uses *comparisons* to decide which operations to perform.

A lower bound for general-purpose sorting

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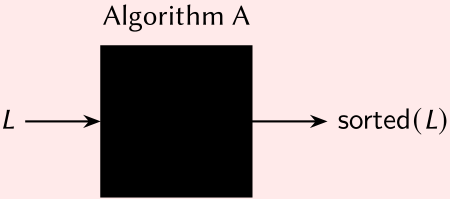

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- Algorithm A uses *comparisons* to decide which operations to perform.

There must be a *distinguishing comparison* after which A behaves *differently*.

A lower bound for general-purpose sorting

We can represent a *distinguishing comparison* via a *comparison tree node*

Consider sorting lists $L[0\ldots,N)$ with values $1,\ldots,N$ in an *unknown order*.

$\mathcal{S}$: All possible lists L that are treated the same by Algorithm A up till this point

C: $L[i] < L[j]$

All lists in $\mathcal{S}$ for which C *did not* hold

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Otherwise not all distinct lists L are processed in a different way.

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We can represent a *distinguishing comparison* via a *comparison tree node*

Consider sorting lists $L[0\ldots, N)$ with values $1, \ldots, N$ in an *unknown order*.

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What is the worst-case length of path π?

The lengths of paths in $\mathcal{T}$ depend on the *height of* $\mathcal{T}$,

$\longrightarrow$ which depends on the *number of leaves* in $\mathcal{T}$.

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- N possible values for the first value,
- $N-1$ possible values for the second value,
- $\ldots$
- 1 possible value for the last value.

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$$\prod_{i=1}^{N} i = N! \text{ leaves} \qquad \text{(all possible permutations).}$$

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Consider sorting lists $L[0\ldots, N)$ with values $1, \ldots, N$ in an *unknown order*.

The *minimal* height of a tree $\mathcal{T}$ with $N!$ leaves

Consider a node n from which we can reach M leaves.

How do we make the distance from n to all its leaves minimal?

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The left and right child of n each can reach $\frac{M}{2}$ leaves:

$\rightarrow$ minimize the size of the tree rooted at *both children*.

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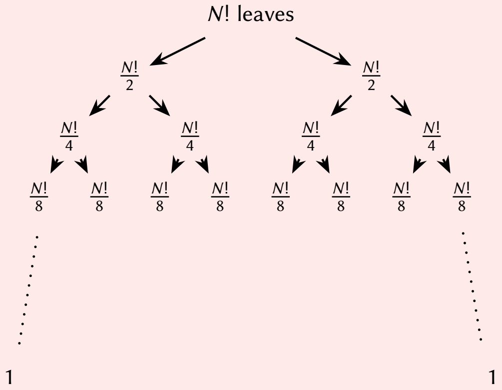

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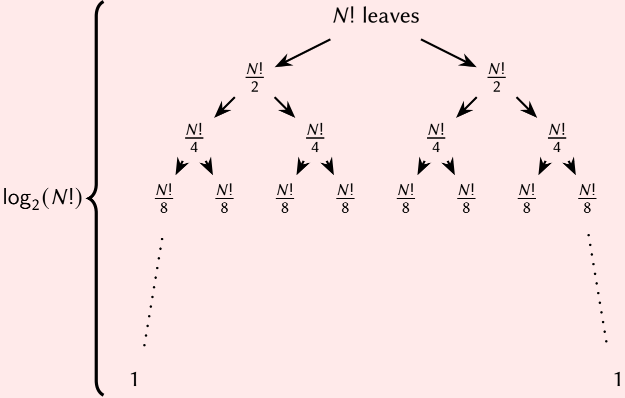

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We have to find a lower bound on $\log_2(N!)$.

$$\log_2(N!) = \log_2(N \cdot (N-1) \cdot \cdots \cdot 1)$$

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\end{aligned}$$

A lower bound for general-purpose sorting

We can represent a distinguishing comparison via a comparison tree node

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&= \tfrac{N}{2}(\log_2(N) - 1) \\
&= \tfrac{N}{2} \log_2(N) - \tfrac{N}{2} = \Theta(N \log_2(N)).
\end{aligned}$$

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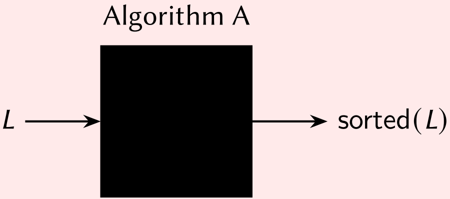
Assume: We have a list $L[0 \ldots N)$ of N distinct values



If Algorithm A is general-purpose, then A will perform *at-least* $\Theta(N \log_2(N))$ *comparisons* for some inputs of N values.

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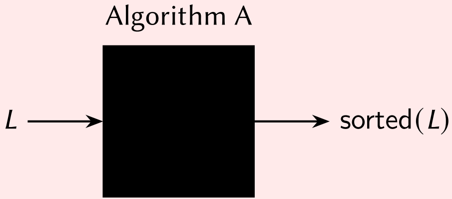

If Algorithm A is general-purpose, then A will perform *at-least* $\Theta(N \log_2(N))$ *comparisons* for some inputs of N values.

If Algorithm A performs less comparisons for *some* inputs, then A will perform more comparisons for *other* inputs.

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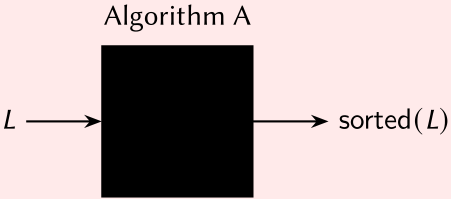

General-purpose sorting algorithms such as MergeSort are *optimal*:
their worst-case complexity matches the lower bound of $\Theta(N \log_2(N))$.

A potentially-faster sort: QuickSort

Can we improve upon the *optimal* MergeSort algorithm?

A potentially-faster sort: QuickSort

Can we improve upon the *optimal* MergeSort algorithm?

- Reduce massive $\Theta(N)$ memory consumption?
- Reduce constants: Merge performs many operations on several lists.

A potentially-faster sort: QuickSort

Divide-and-conquer

Divide Turn problem into smaller subproblems.

Conquer Solve the smaller subproblems using *recursion*.

Combine Combine the subproblem solutions into a final solution.

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Dividing a list into *small* and *large* values sounds easier than Merge!

QuickSort: High-level overview

Algorithm QuickSort($L[start \ldots end]$):

if $end - start > 1$ **then**

2	6	3	5	1	4

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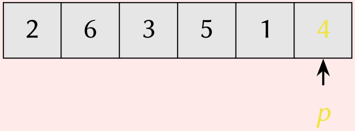

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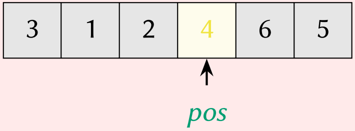

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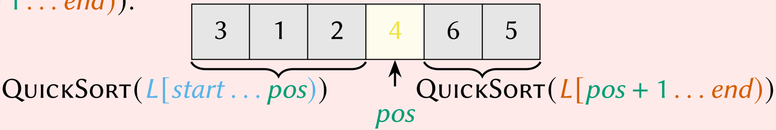

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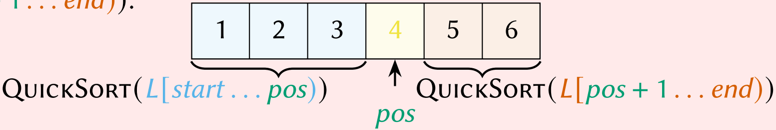

Proof of correctness: QuickSort($L[start \ldots end]$) sorts

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Base case QuickSort sorts $0 \leq end - start \leq 1$ values.

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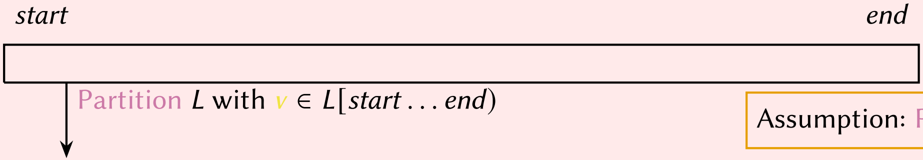

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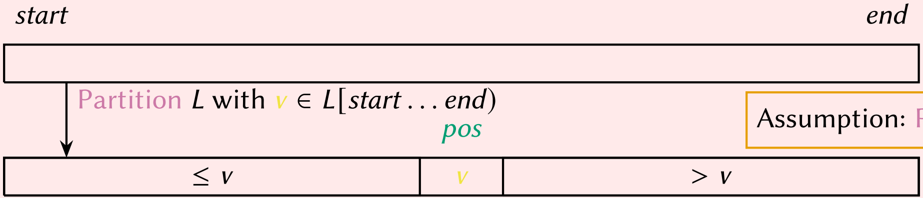

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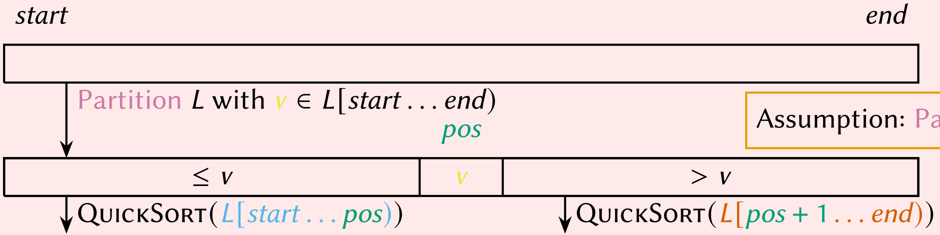

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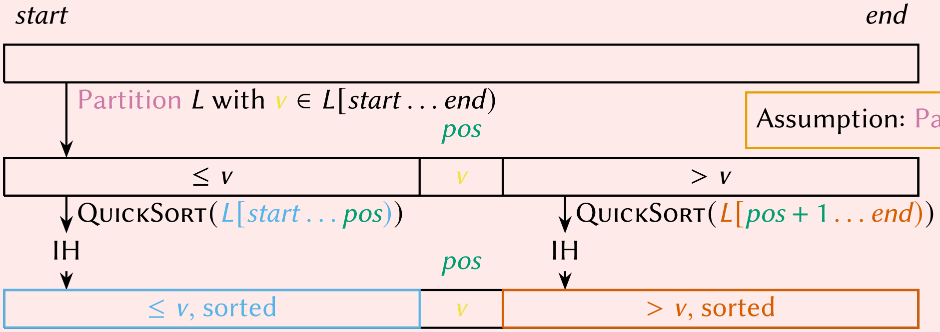

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$pos < end$

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Algorithm PARTITION(L, $start$, end, p):

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Algorithm PARTITION(L, *start*, *end*, p):

1: Exchange $L[start]$ and $L[p]$.

Assumption: Partition is correct

Algorithm PARTITION(L, $start$, end, p):

1: Exchange $L[start]$ and $L[p]$.
2: $v, i, j := L[start], start, start + 1$.

Values in $L[start + 1 \dots i + 1)$ are smaller-or-equal to v.

Values in $L[i + 1 \dots j)$ are larger than v.

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Algorithm PARTITION(L, *start*, *end*, p):

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$v, i, j := L[start], start, start + 1$.

Values in $L[start + 1 \dots i + 1)$ are smaller-or-equal to v.

Values in $L[i + 1 \dots j)$ are larger than v.

Exchange $L[i]$ and $L[start]$.

return i.

Assumption: Partition is correct

Algorithm PARTITION(L, $start$, end, p):

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Exchange L[start] and L[p].
v, i, j := L[start], start, start + 1.
Values in L[start + 1 ... i + 1) are smaller-or-equal to v.
Values in L[i + 1 ... j) are larger than v.
while j ≠ end do
    if L[j] ≤ v then
        i := i + 1.
        Exchange L[i] and L[j].
    j := j + 1.
Exchange L[i] and L[start].
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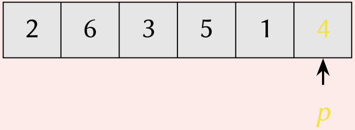

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Values in $L[i + 1 \dots j)$ are larger than v.
while $j \neq end$ **do**
  if $L[j] \leq v$ **then**
    $i := i + 1$.
    Exchange $L[i]$ and $L[j]$.
  $j := j + 1$.
Exchange $L[i]$ and $L[start]$.
return i.

4	6	3	5	1	2

Assumption: Partition is correct

Algorithm PARTITION(L, *start*, *end*, p):

```
Exchange L[start] and L[p].
v, i, j := L[start], start, start + 1.
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```

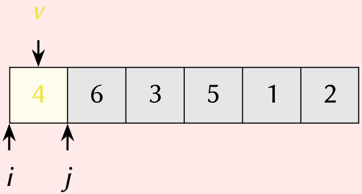

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        Exchange L[i] and L[j].
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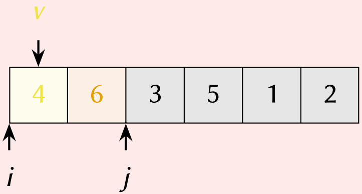
```



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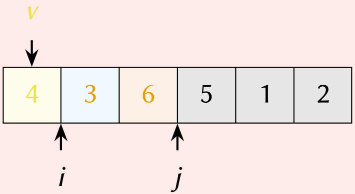

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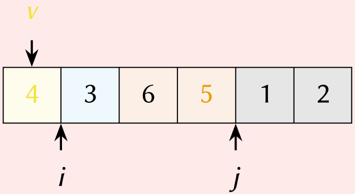

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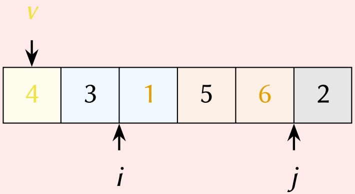

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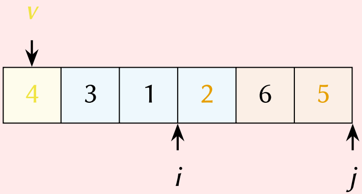

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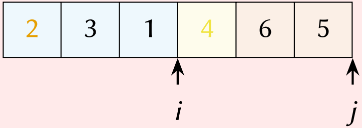

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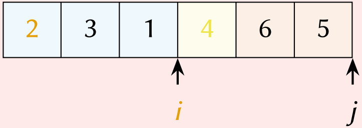


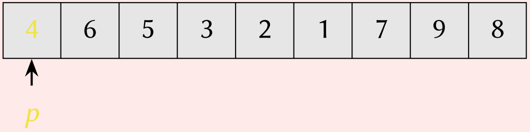
QuickSort: A complete example

We did not specify yet how to choose a pivot value!

4	6	5	3	2	1	7	9	8

QuickSort: A complete example

We did not specify yet how to choose a pivot value → random choices for now.



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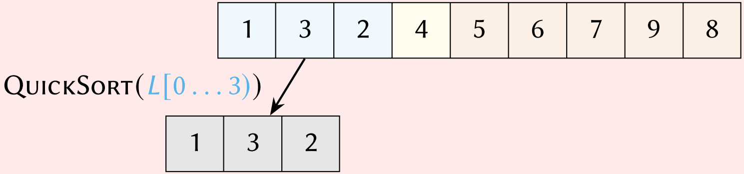

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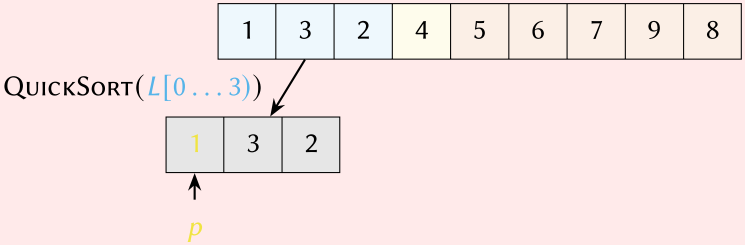

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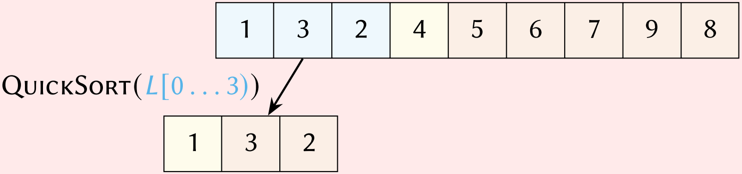

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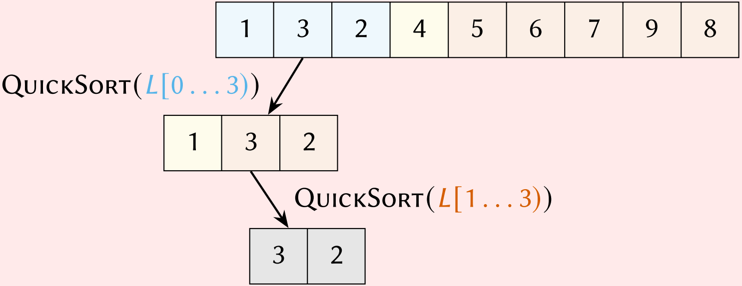

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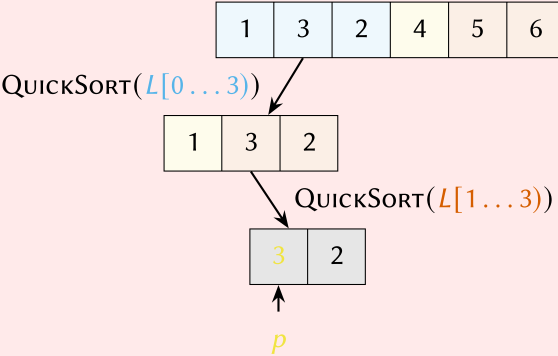

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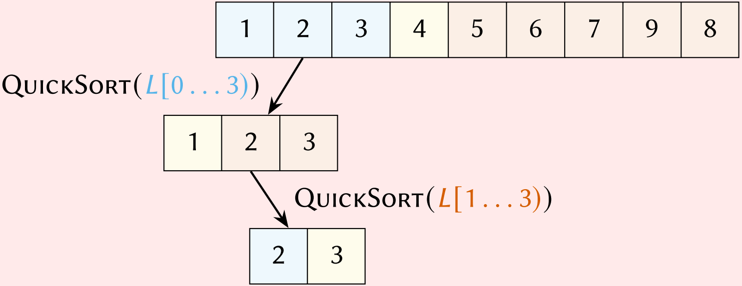

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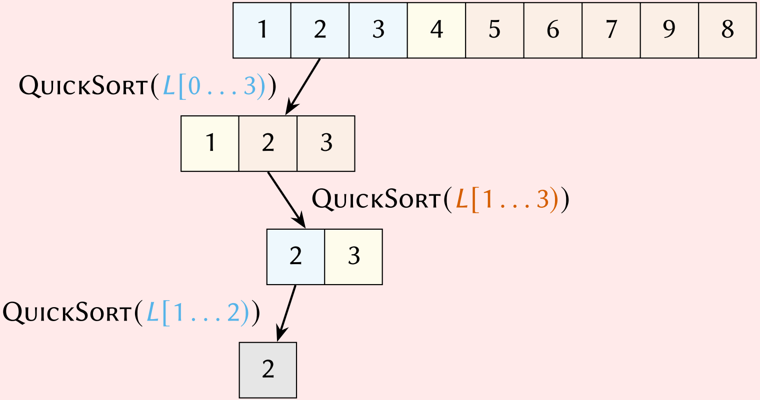

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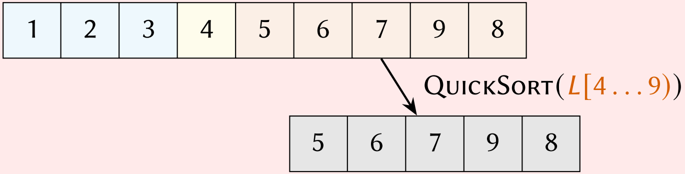

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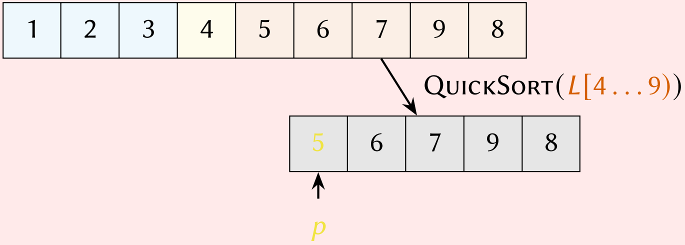

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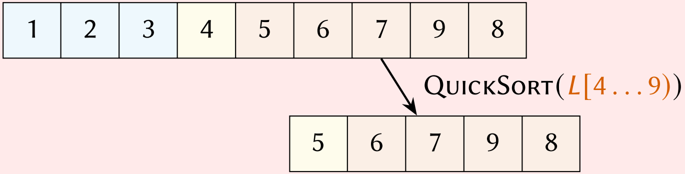

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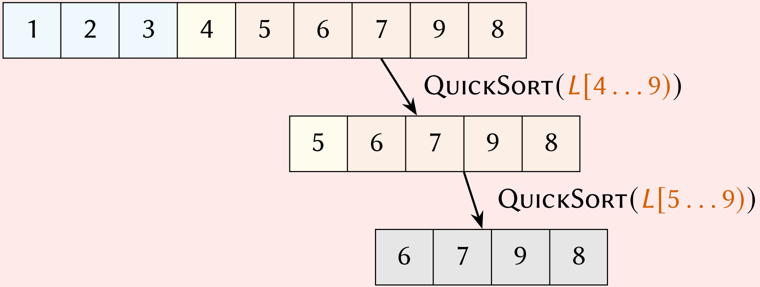

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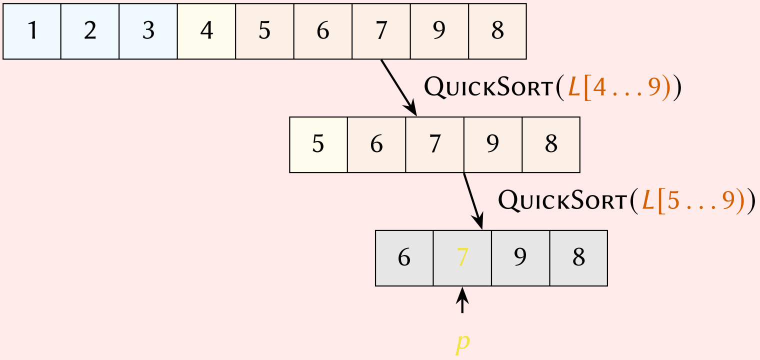

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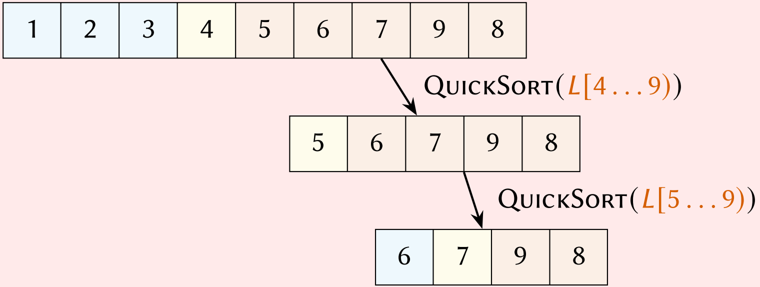

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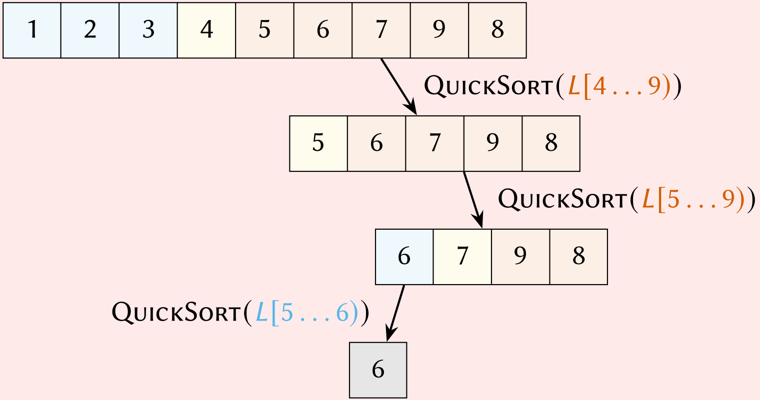

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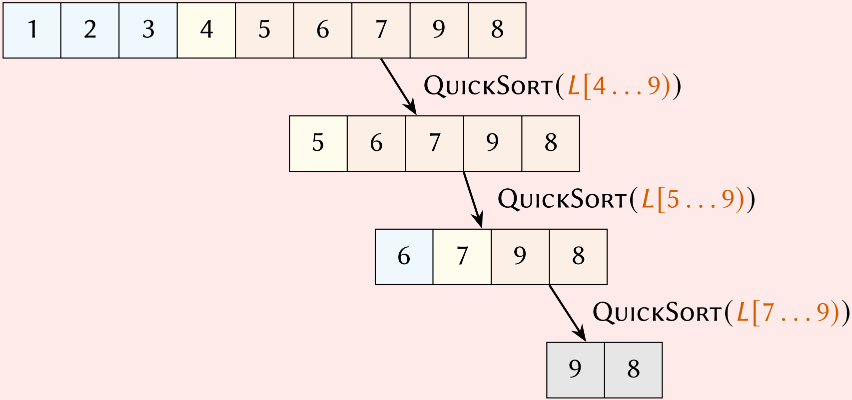

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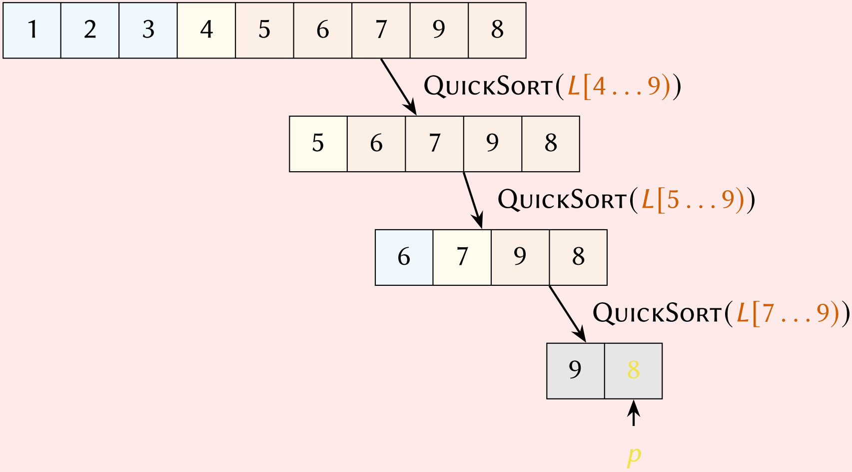

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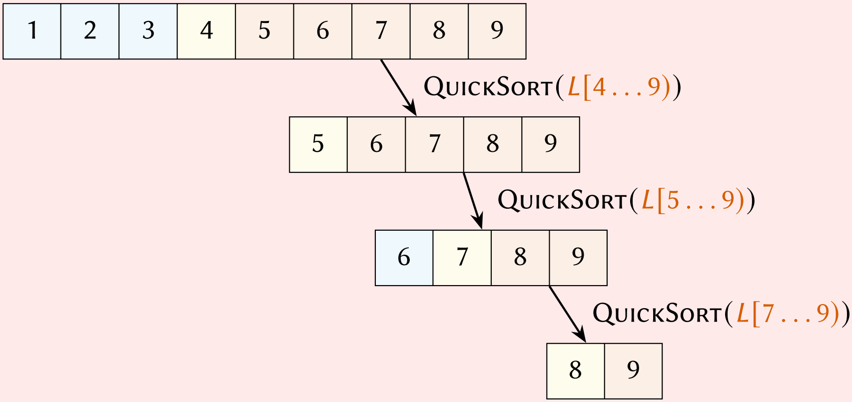

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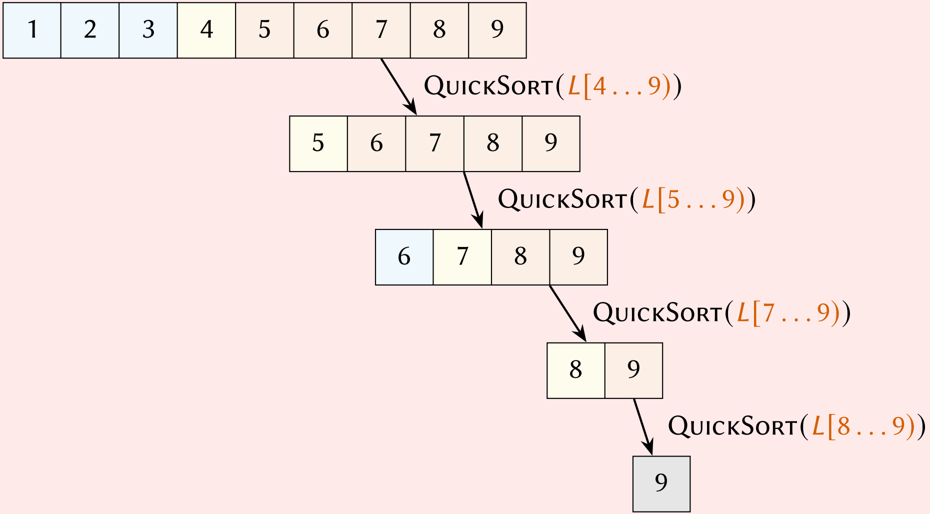

QuickSort: A complete example

We did not specify yet how to choose a pivot value → random choices for now.

1	2	3	4	5	6	7	8	9

The complexity of QuickSort

The complexity of QuickSort

The complexity of QuickSort depends on the chosen pivot values.

The complexity of QuickSort

Example: Pivots are always smaller than all other values

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ & \text{if } N > 1. \end{cases}$$

The complexity of QuickSort

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$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ T(N-1) + N & \text{if } N > 1. \end{cases}$$

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	Number	Cost	Total
N	1	N	N

The complexity of QuickSort

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↓			
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↓			
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↓			
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↓			
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↓			
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↓			
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↓			
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$\vdots$	$\vdots$	$\vdots$	$\vdots$
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$\downarrow$			
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$\downarrow$			
$N-2$	1	$N-2$	$N-2$
$\downarrow$			
$N-3$	1	$N-3$	$N-3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	1	1

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2} = \Theta(N^2).$$

The complexity of QuickSort

Example: Pivots are "in the middle" of all values

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ & \text{if } N > 1. \end{cases}$$

The complexity of QuickSort

Example: Pivots are “in the middle” of all values

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T(\lfloor \frac{N}{2} \rfloor) + N & \text{if } N > 1. \end{cases}$$

The complexity of QuickSort

Example: Pivots are “in the middle” of all values

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T(\lfloor \frac{N}{2} \rfloor) + N & \text{if } N > 1. \end{cases}$$

We have seen this one before: $T(N) = \Theta(N \log_2(N))$.

The complexity of QuickSort

The complexity of QuickSort depends *a lot* on the chosen pivot values.

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Randomized QuickSort: Choose pivot values fully at random

We *cannot* provide an exact complexity for Randomized QuickSort:
Executions on *the same list* can have vastly different random choices (and complexities).

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Expected-case analysis: an analysis in terms of the distribution of random choices.

Expected-case analysis is *not* average-case analysis!
Average-case analysis: an analysis in terms of the distribution of inputs.

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Randomized QuickSort: Choose pivot values fully at random

We *cannot* provide an exact complexity for Randomized QuickSort:
Executions on *the same list* can have vastly different random choices (and complexities).

Expected-case analysis: an analysis in terms of the distribution of random choices.

Any random choice in Randomized QuickSort is equally likely:

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ \frac{1}{N}\left(\sum_{i=0}^{N-1}\Big(T(i) + T(N-(i+1))\Big)\right) + N & \text{if } N > 1. \end{cases}$$

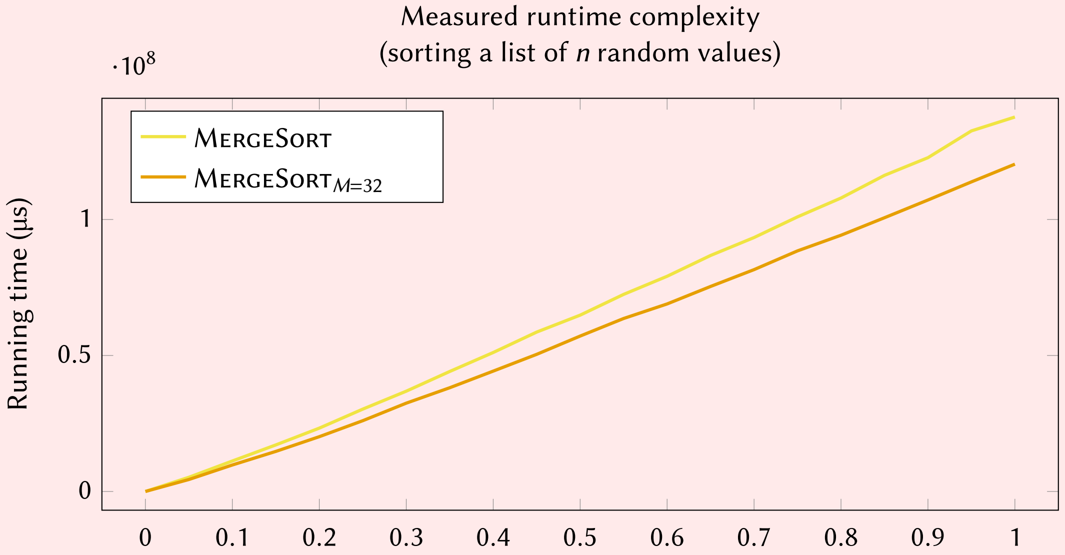
With some *mathematical tricks*, we can show that $T(N) = \Theta(N \log_2(N))$.

The complexity of QuickSort

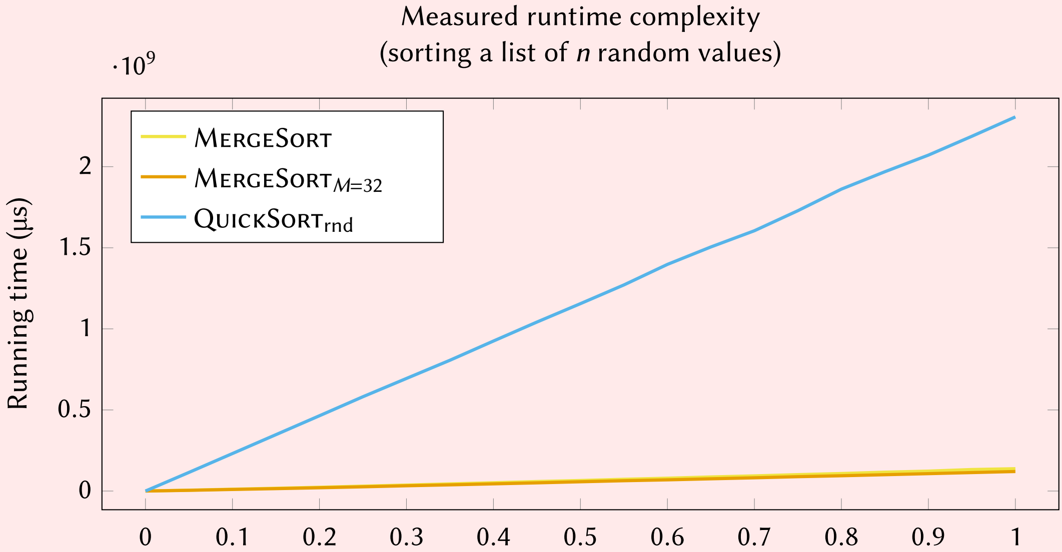
The complexity of QuickSort depends *a lot* on the chosen pivot values.

We will later develop a QuickSort variant that always has a $\Theta(N \log_2(N))$ complexity, this independent of how pivot values are chosen.

The performance of QuickSort



The performance of QuickSort



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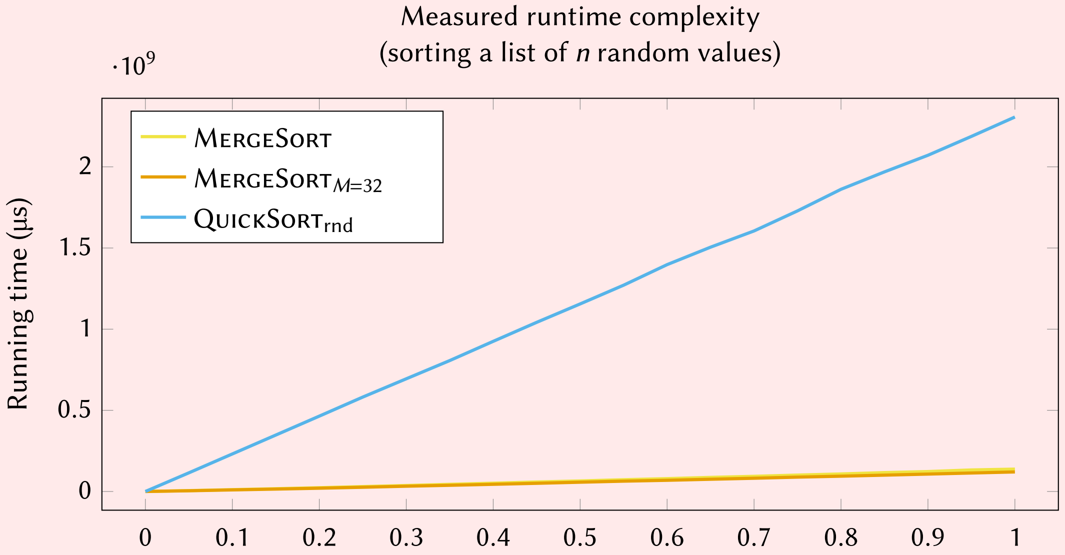

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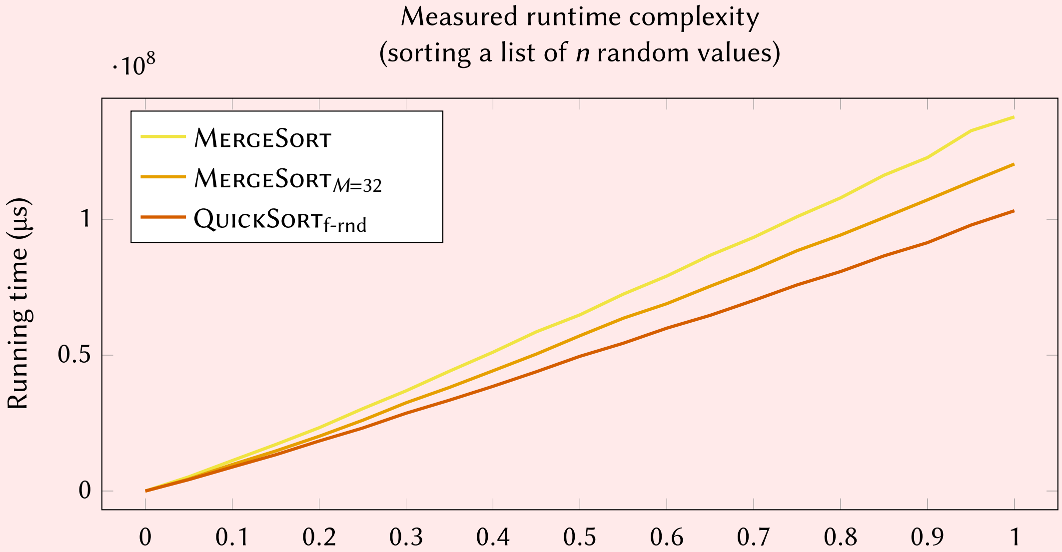

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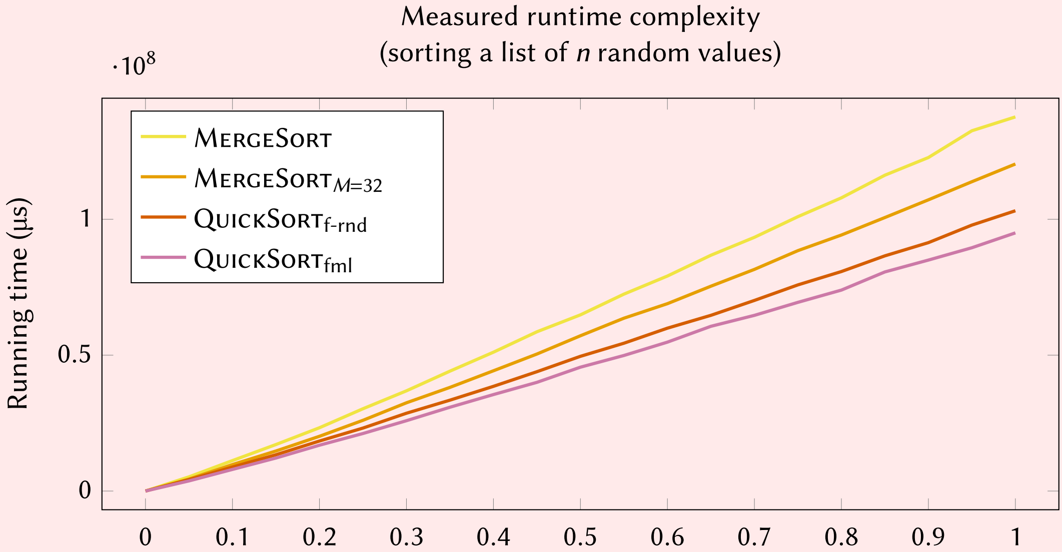

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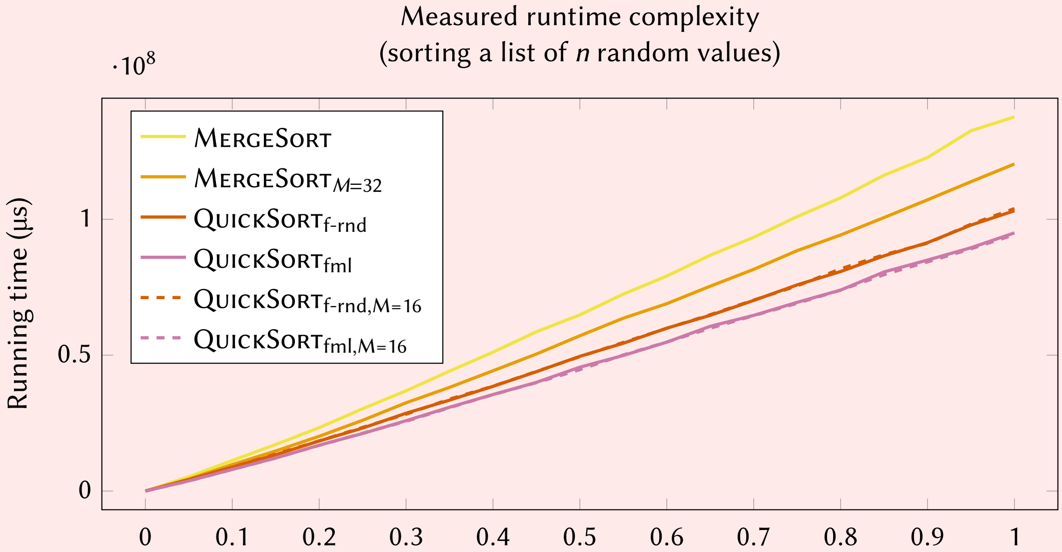

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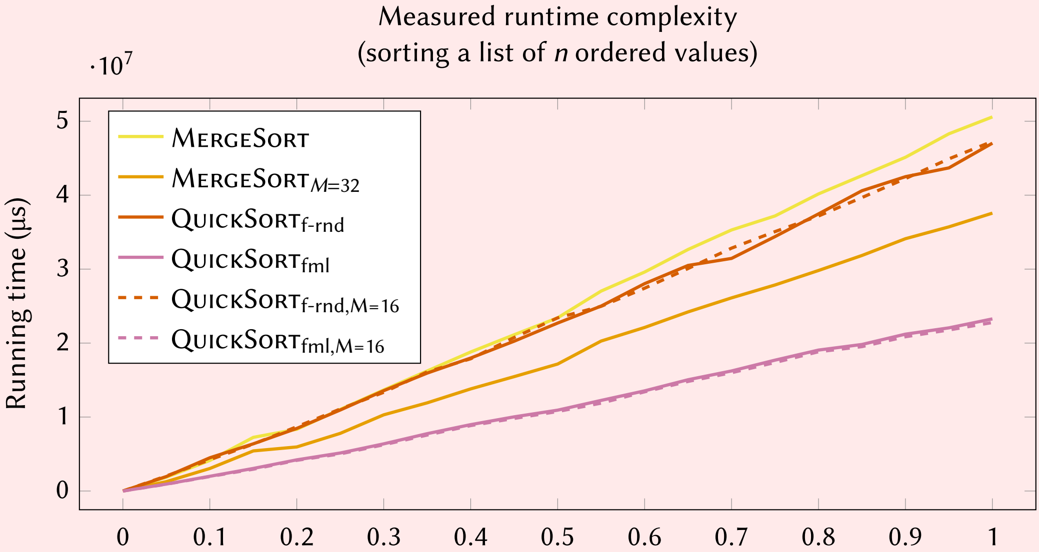

Further comparing MergeSort and QuickSort

	Comparisons	Changes	Memory
MergeSort	$\Theta(N \log_2(N))$	$N \log_2(N)$	$\Theta(N)$
QuickSort	$\Theta(N \log_2(N))$ (expected)	$\Theta(N \log_2(N))$ (expected)	$\Theta(\log_2(N))$ (expected)

Further comparing MergeSort and QuickSort

QuickSort is *not* stable

Consider a L list of pairs $(name, age)$ that is already sorted on age:

$$L = [(\text{Alicia}, 12), (\text{Dafni}, 20), (\text{Celeste}, 27), (\text{Dafni}, 35), (\text{Alicia}, 56), (\text{Celeste}, 80)].$$

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- QuickSort$(L[0, 6))$ on names only will *not maintain* ordering on age:

$$[(\text{Alicia}, 56), (\text{Alicia}, 12), (\text{Celeste}, 27), (\text{Celeste}, 80), (\text{Dafni}, 35), (\text{Dafni}, 20)].$$

Further comparing MergeSort and QuickSort

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- QuickSort($L[0, 6)$) on names only will *not maintain* ordering on age:

  $$[(\text{Alicia}, 56), (\text{Alicia}, 12), (\text{Celeste}, 27), (\text{Celeste}, 80), (\text{Dafni}, 35), (\text{Dafni}, 20)].$$

- MergeSort($L[0, 6)$) on names only will *always maintain* pre-existing ordering (for values that are “identical”):

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Further comparing MergeSort and QuickSort

QuickSort is *not* stable

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We say that MergeSort is *stable*.

Using Partition: Order statistics

Problem

Given a list $L[start \ldots end)$ and k, $start \leq k < end$, return the k-th smallest value in $L[start \ldots end)$.

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Algorithm SELECT(L, $start$, end, k):

```
Choose the position p ∈ [start, end) of the pivot value v := L[pos].
pos := PARTITION(L, start, end, p).
if pos = k then
    return L[pos].
else if pos > k then
    return SELECT(L, start, pos − 1, k).
else
    return SELECT(L, pos, end, k).
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    return SELECT($L, start, pos - 1, k$).
else
    return SELECT(L, pos, end, k).

Essentially a "half" QUICKSORT that only sorts those values that could be the k-th.

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Randomized SELECT: $\Theta(N)$ (expected).

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Select(L, 0, 9, 6): We want the $k = 6$-th smallest value.

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↑

p

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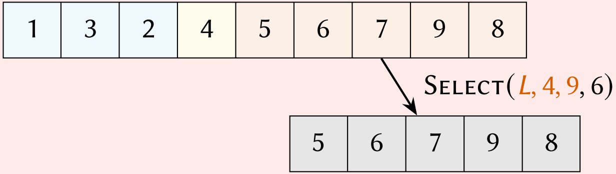

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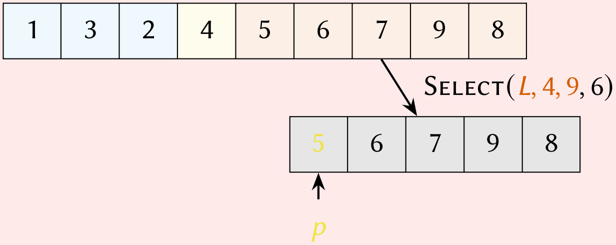

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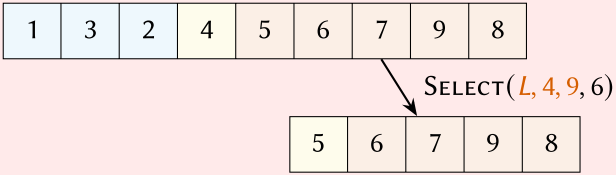

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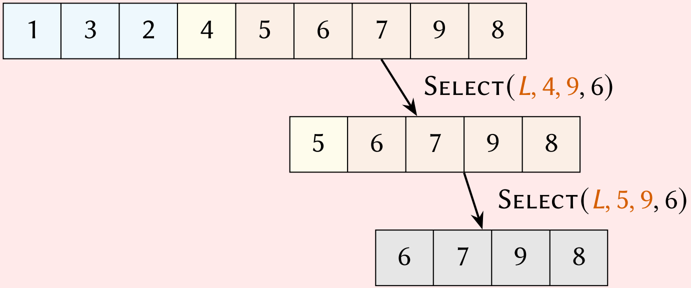

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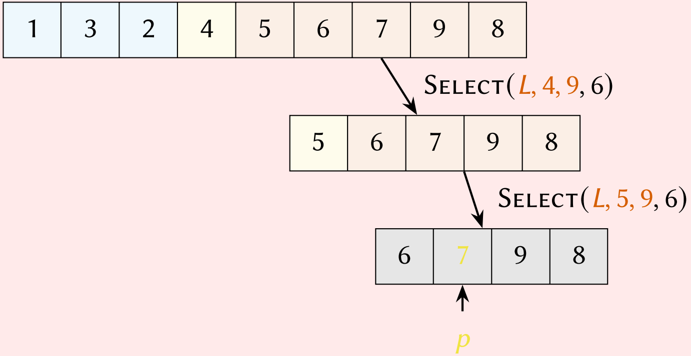

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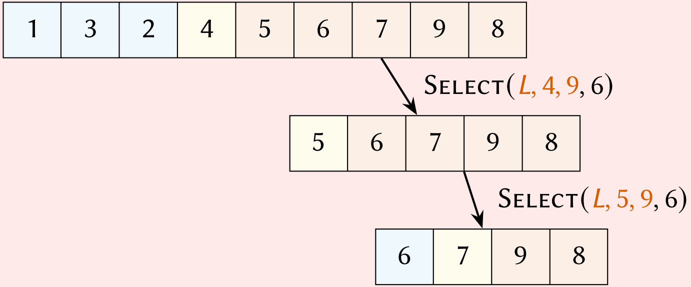

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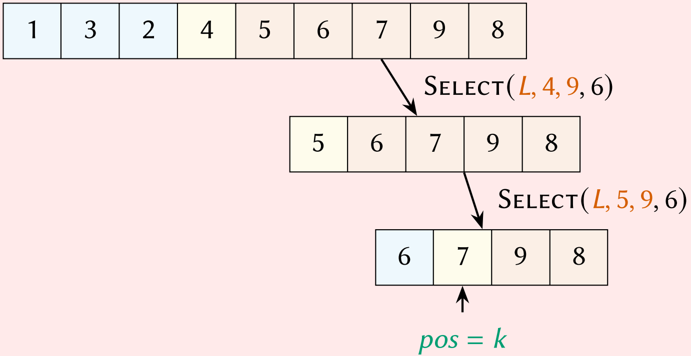

Final notes on QuickSort

	C++	Java
QuickSort	std::sort	java.util.Arrays.sort (non-Objects)
Partition	std::partition	
(related)	std::stable_partition	