Strings

SFWRENG 2CO3: Data Structures and Algorithms

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Winter 2024

Strings over alphabets

An *alphabet* $\mathcal{A}$ is a finite set of distinct symbols.

A *string* over $\mathcal{A}$ is a sequence of symbols taken from $\mathcal{A}$.

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Examples

- A typical string over the roman alphabet {'a', . . . , 'z', '␣'}:
  "hello", "hello␣world", "strings␣over␣alphabets", and "".
- A *bit string* is a sequence over $\{0, 1\}$:
  "0011", "1101100", "", and "0".
- A *DNA String* is a sequence over $\{A, C, G, T\}$:
  "", "AACATG", "AGT", and "AAACCCAAATTT".
- A *Unicode string* is a sequence over the unicode code points
  (149 186 symbols and counting).
- A *byte string* is a sequence over *bytes*.

Operations on strings and alphabets

We assume the following basic operations:

- We can sequentially iterate over the symbols in a string.
- We can look up the i-th symbol in a string in $\Theta(1)$.
  (This can be hard in some practical settings: UTF-8 and UTF-16 strings do *not* support this).
- We assume that each alphabet $\mathcal{A}$ is an ordered list L of symbols.
- For each $\sigma \in \mathcal{A}$, we can determine its position in L.

BucketSort: A special-purpose sort

Assumption We have M distinct symbols with values in the range $0, \ldots, M-1$.

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for all $v \in L$ **do**
  $buckets[v] := buckets[v] + 1$.   } $\Theta(|L|)$
$k := 0$.
for all $i := 0$ upto $M-1$ **do**
  for all $j := 0$ upto $buckets[i]$ **do**
    $L[k] := i$.   } $|L|$
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Lines 1–3: $\Theta(|L|)$

Lines 6–8: $|L|$

Lines 5–8: $\Theta(M + |L|)$

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$\Theta(M)$ (first line)

$\Theta(|L|)$ (the first **for all** loop)

$|L|$ (the inner **for all** loop)

$\Theta(M + |L|)$ (the outer **for all** loop)

$\Theta(M + |L|)$ (the whole algorithm)

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$L =$ "GAGGATATGTAG".

A:	0
C:	0
G:	0
T:	0

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C:	0
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$L =$ "GAGGATATGTAG".

A:	1
C:	0
G:	2
T:	0

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A:	1
C:	0
G:	3
T:	0

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A:	2
C:	0
G:	3
T:	0

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A:	3
C:	0
G:	4
T:	2

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L = “GAGGATATGTAG”.

A:	4
C:	0
G:	5
T:	3

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Algorithm GBucketSort(L, r):

```
buckets := [[ ] | 0 ≤ i ≤ M − 1].
for all v ∈ L do
    Append v to buckets[r(v)].
k := 0.
for all i := 0 upto M − 1 do
    for all j := 0 upto |buckets[i]| do
        L[k] := buckets[i][j].
        k := k + 1.
```

Generalization

Assume we have values that "represent" $0, \ldots, M-1$ via some function r.

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Generalization

Assume we have values that "represent" $0, \ldots, M-1$ via some function r.

Notice that GBucketSort is *stable*.

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Assumption. We have strings of length k over alphabet $\mathcal{A}$.

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for d := k − 1 downto 0 do
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(lines 2–3: $\Theta(|L| + |\mathcal{A}|)$)

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$$\left.\begin{array}{l}\text{Stable-sort } L \text{ on the } d\text{-th string symbols.} \\ \text{GBucketSort}(L, r_d) \text{ with } r_d(S) = S[d].\end{array}\right\} \Theta(|L| + |\mathcal{A}|)$$

Whole loop: $\Theta(k(|L| + |\mathcal{A}|))$

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L = ["AGCTCT",
"ATTAAC",
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$\longrightarrow$

L = ["AGCTGA", "ATCTAA", "GTCTGC", "ATTAAC", "GCGCGG", "GGCGCG", "TCTATG", "TCACCG", "TGGACG", "AGCTCT"]

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$\longrightarrow$

L = [“ATCTAA”, “ATTAAC”, “GGCGCG”, “TCACCG”, “TGGACG”, “AGCTCT”, “AGCTGA”, “GTCTGC”, “GCGCGG”, “TCTATG”]

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L = [“ATCTAA”, “ATTAAC”, “GGCGCG”, “TCACCG”, “TGGACG”, “AGCTCT”, “AGCTGA”, “GTCTGC”, “GCGCGG”, “TCTATG”]

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L = [“ATTAAC”, “TGGACG”, “TCTATG”, “TCACCG”, “GCGCGG”, “GGCGCG”, “ATCTAA”, “AGCTCT”, “AGCTGA”, “GTCTGC”]

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L = ["ATTAAC", "TGGACG", "TCTATG", "TCACCG", "GCGCGG", "GGCGCG", "ATCTAA", "AGCTCT", "AGCTGA", "GTCTGC"]

→

L = ["TCACCG", "GGCGCG", "ATCTAA", "AGCTCT", "AGCTGA", "GTCTGC", "TGGACG", "GCGCGG", "ATTAAC", "TCTATG"]

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"GTCTGC",	→	"AGCTGA",
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"GCGCGG",		"ATCTAA",
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L = [“TCACCG”, “GCGCGG”, “TCTATG”, “GGCGCG”, “AGCTCT”, “AGCTGA”, “TGGACG”, “ATCTAA”, “GTCTGC”, “ATTAAC”]

→

L = [“AGCTCT”, “AGCTGA”, “ATCTAA”, “ATTAAC”, “GCGCGG”, “GGCGCG”, “GTCTGC”, “TCACCG”, “TCTATG”, “TGGACG”]

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Correctness

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Invariant: In L, the suffix of the last $k - (d + 1)$ symbols is sorted.

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Generalization: strings with variable lengths up-to-k

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Correctness

Invariant: In L, the suffix of the last $k - (d + 1)$ symbols is sorted.

Generalization: strings with variable lengths up-to-k

Let S be a string of length $|S| < k$.

Interpret $S[|S|], \ldots, S[k-1]$ as symbols that come before all other symbols.

The book calls this *least-significant-digit string sort*.

RadixSort: A special-purpose sort for strings

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Algorithm RadixSort(L):

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$\Theta(k(|L| + |\mathcal{A}|))$ versus $\Theta(k|L|\log(|L|))$ (or $\Theta(|L|\log^2(|L|))$ expected).

Most-significant-digit string sort

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After *constructing* the buckets with respect to r_0,
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Algorithm MSD-Sort(L, d):

1: **if** $d < k$ **and** $|L| > 1$ **then**
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during which we further sort each
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    $buckets := [[\,] \mid 0 \le i \le |\mathcal{A}| - 1]$.
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Finetuning

We end up with many arrays *buckets* that each hold $|\mathcal{A}|$ lists!

Most-significant-digit string sort

Algorithm MSD-SORT(L, d):

```
if |L| ≤ |𝒜| then
    Use another algorithm to sort L.
else if d < k and |L| > 1 then
    buckets := [[ ] | 0 ≤ i ≤ |𝒜| − 1].
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Algorithm MSD-SORT(L, d):

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Complexity

At-most $\Theta(k(|L| + |\mathcal{A}|))$.

Sorting: Best practices

So which sort algorithm is the best?

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Depends on the to-be-sorted input.

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Often, your standard sort algorithm will be sufficient.

Tries: special-purpose sets and dictionaries

Assumption We have an alphabet $\mathcal{A}$ with $M = |\mathcal{A}|$ symbols.

A *Trie* is a set representation that can hold strings over $\mathcal{A}$ such that:

- strings of length N can be *added* in $\Theta(N)$;
- strings of length N can be *removed* in $\Theta(N)$;
- *checking* whether a string of length N is in the set costs $\Theta(N)$;
- one can efficiently *iterate* over all strings in the set (in sorted order).

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We have seen tries with $\mathcal{A} = \{0, 1\} \rightarrow$ BSSet in Example Assignment 3.

Tries: special-purpose sets and dictionaries

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Each node n in a *trie* T over $\mathcal{A} = \{\sigma_1, \dots, \sigma_M\}$ has

- a flag $n.end$ that is `true` if the node n represents a string in T;
- at-most M edges to children labeled $\sigma_1, \dots, \sigma_M$.

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Each node n in a *trie* T over $\mathcal{A} = \{\sigma_1, \dots, \sigma_M\}$ has

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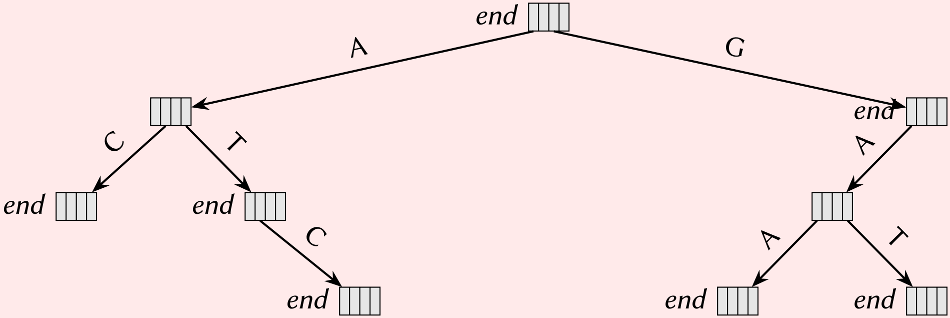
- at-most M edges to children labeled $\sigma_1, \dots, \sigma_M$.

Tries: special-purpose sets and dictionaries

Assumption We have an alphabet $\mathcal{A}$ with $M = |\mathcal{A}|$ symbols.



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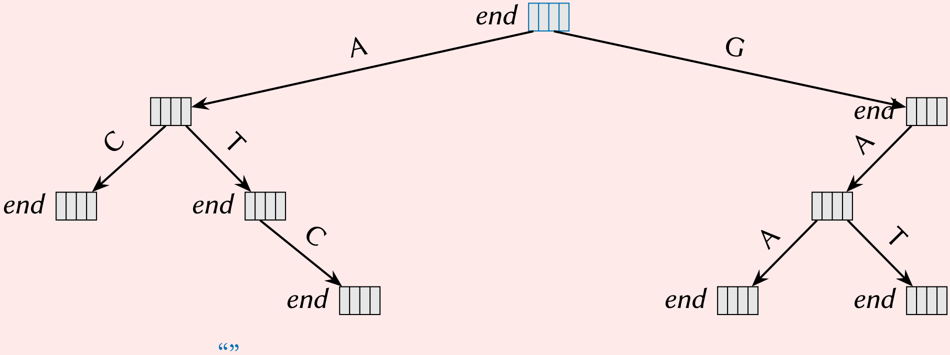

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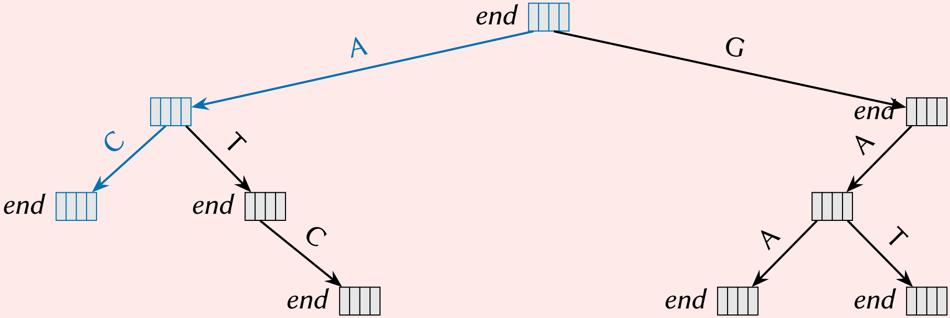

“”, “AC”

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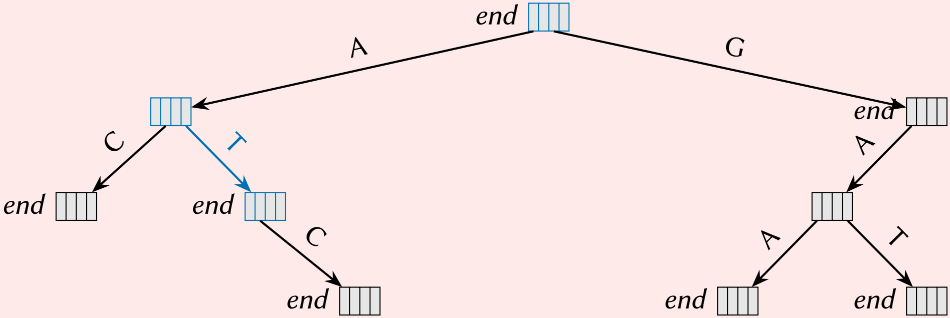

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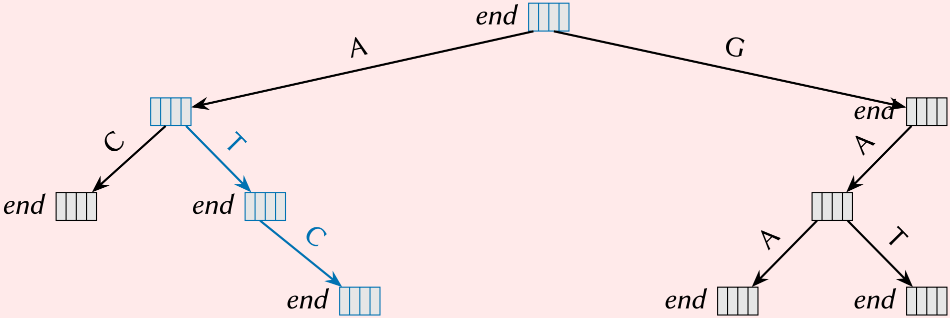

“”, “AC”, “AT”, “ATC”

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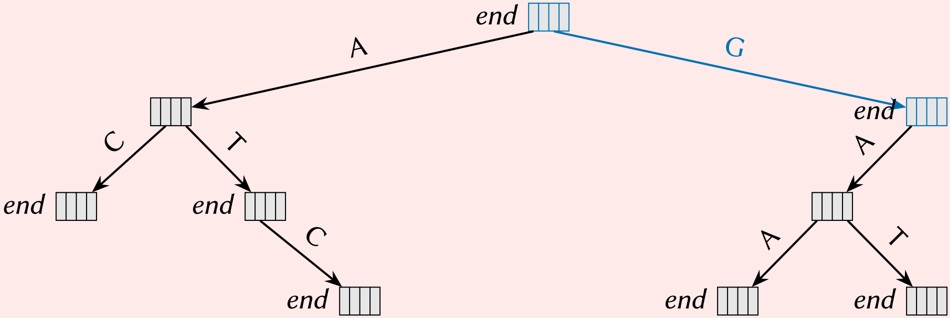

"", "AC", "AT", "ATC", "G"

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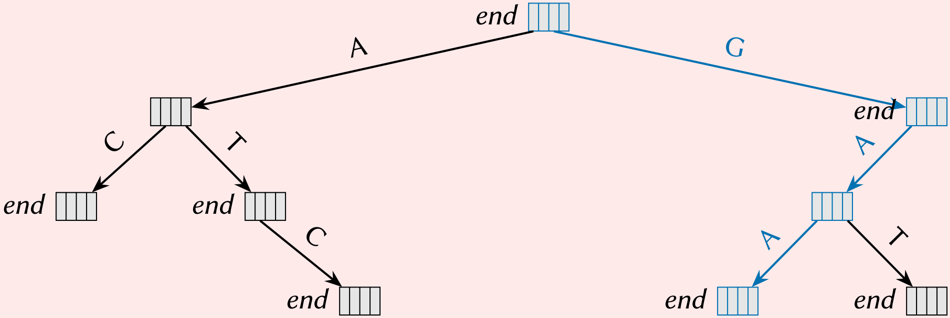

"", "AC", "AT", "ATC", "G", "GAA"

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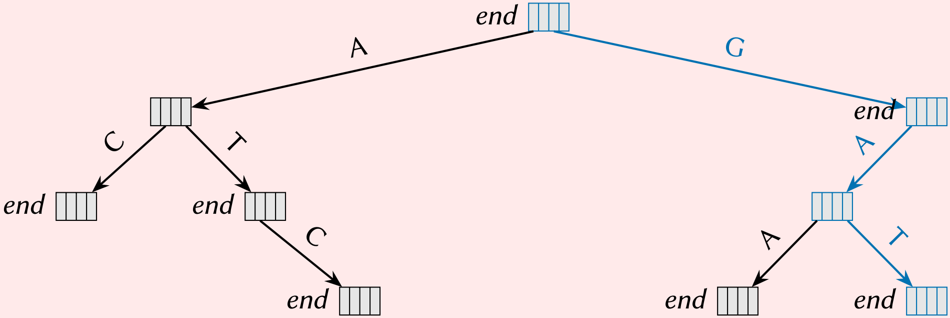

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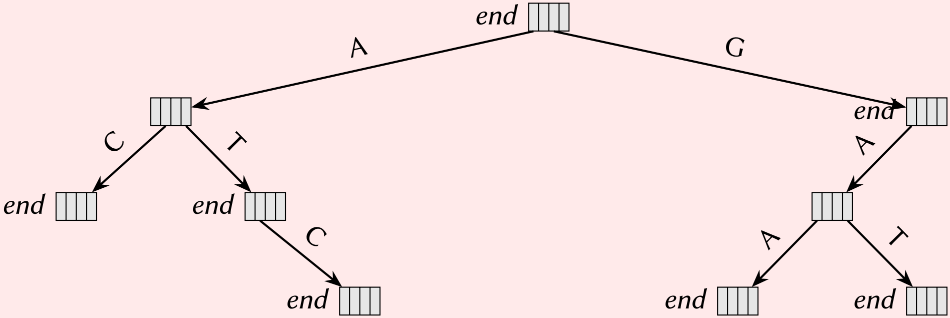

Adding a string

- Follow-or-make a path according to the string symbols.
- Set *n.end* on the last node *n* on this path.

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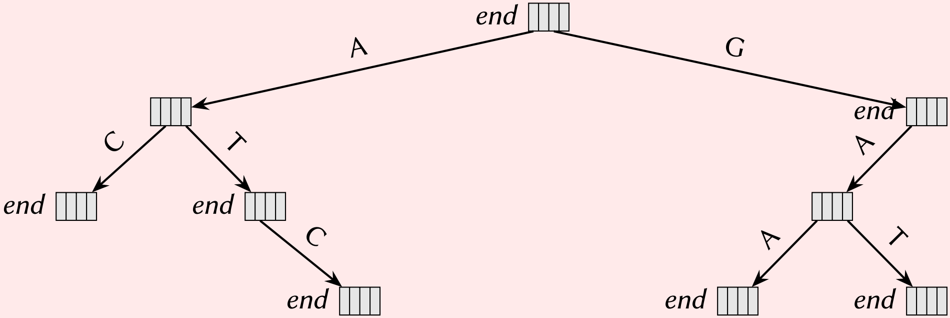

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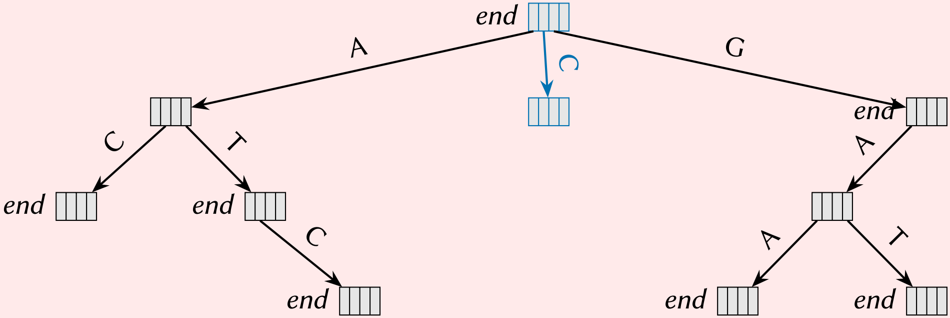

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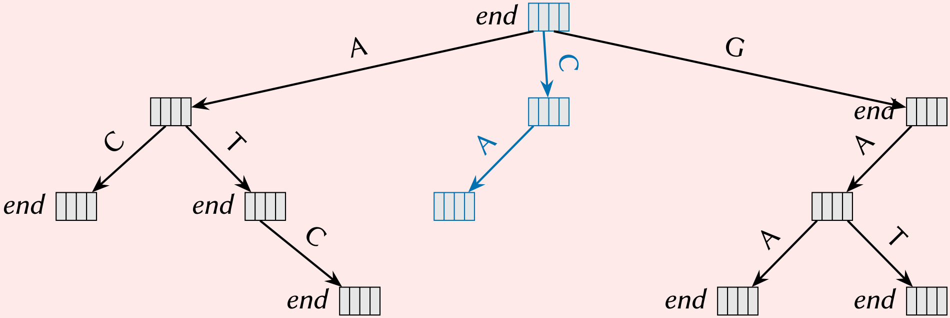

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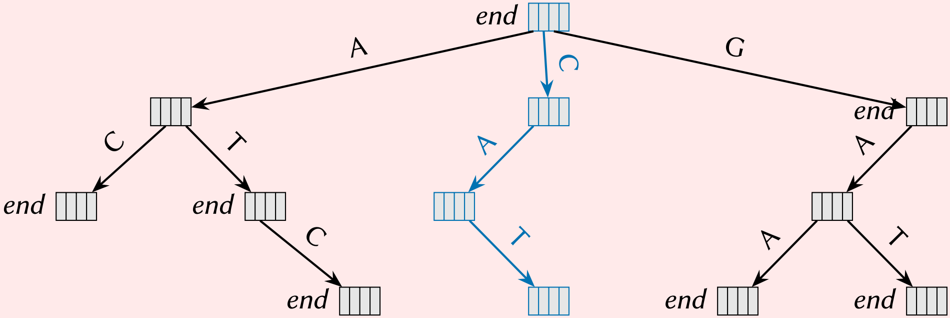

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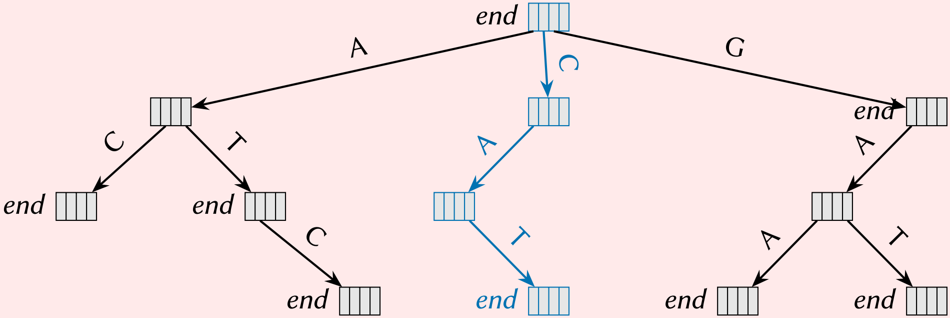

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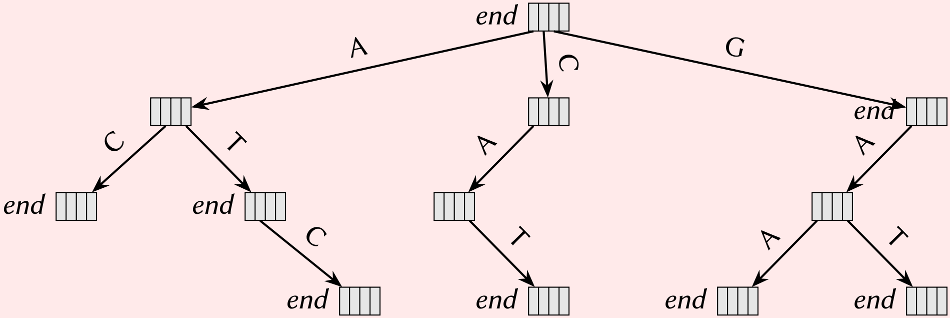

Adding a string "GA"

- Follow-or-make a path according to the string symbols.
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Tries: special-purpose sets and dictionaries

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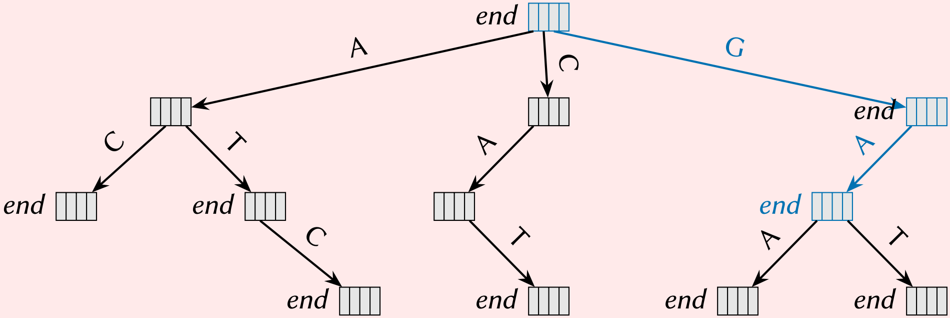

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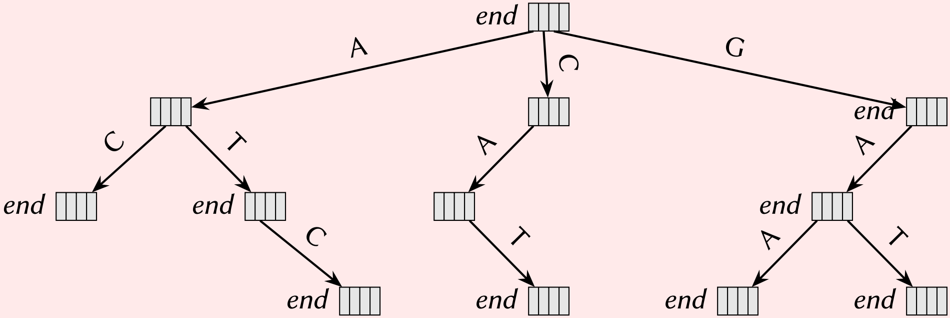

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- Follow a path according to the string symbols to node n and unset $n.end$.
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Tries: special-purpose sets and dictionaries

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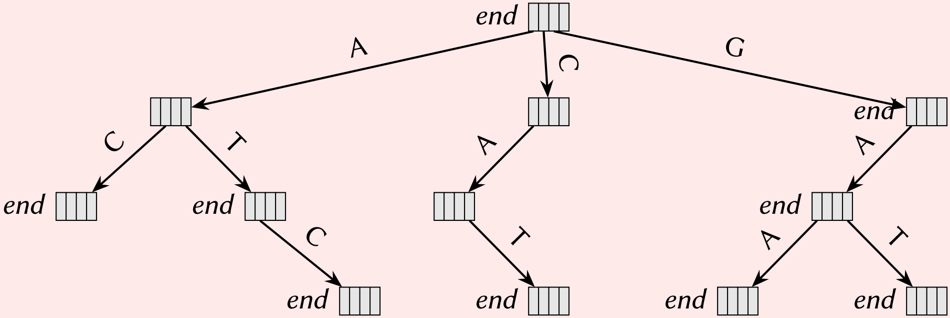

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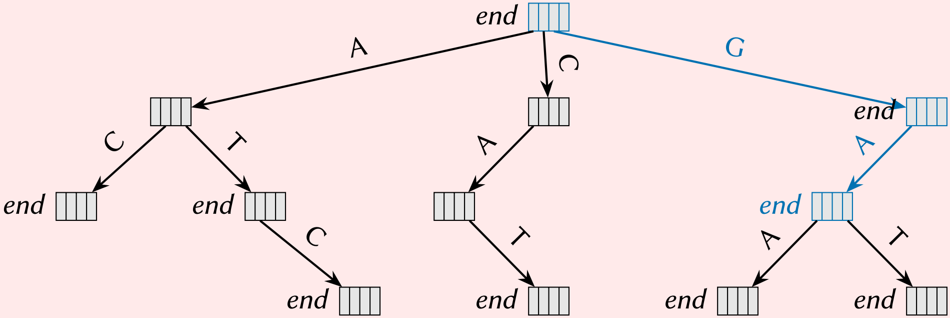

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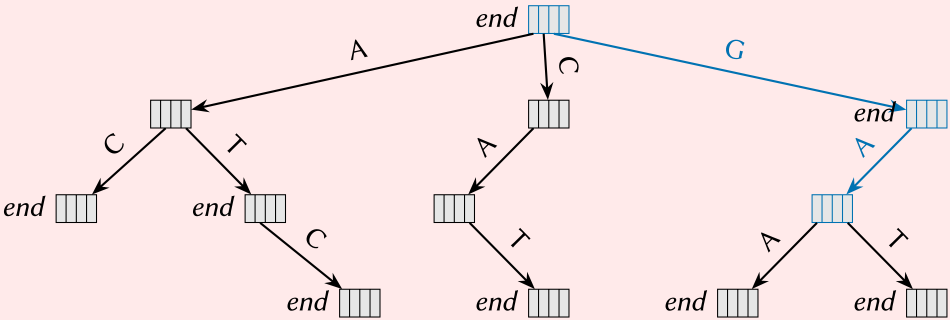

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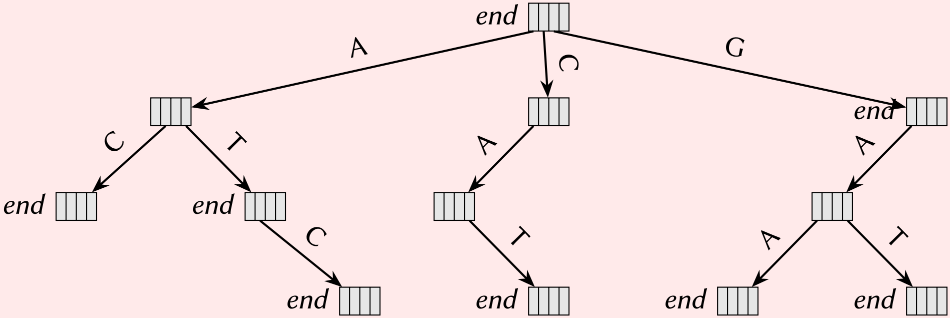

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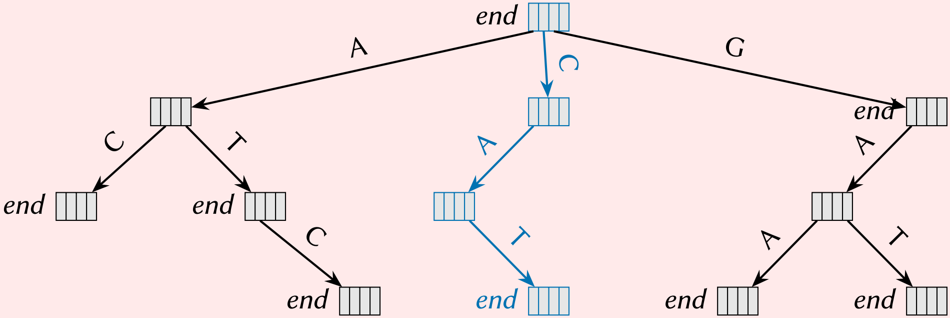

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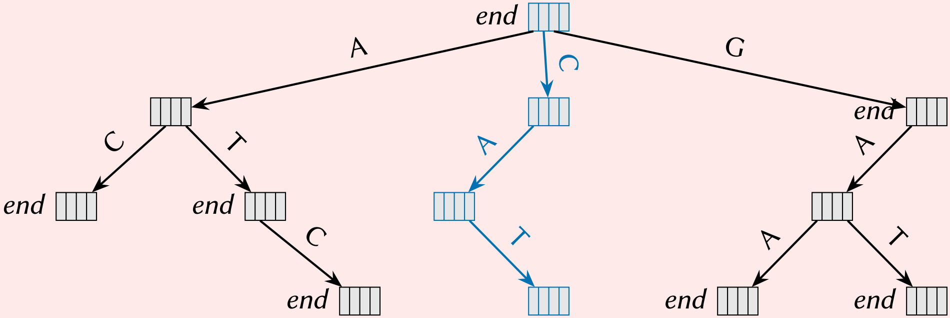

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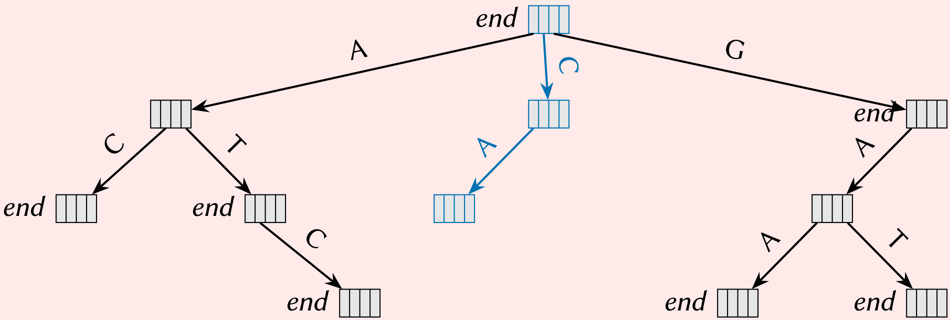

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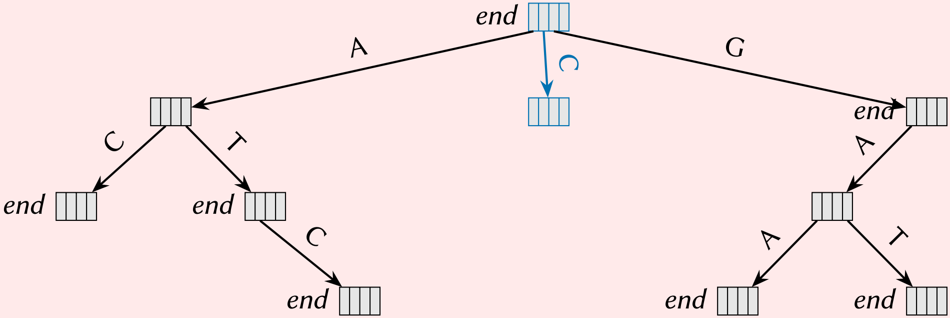

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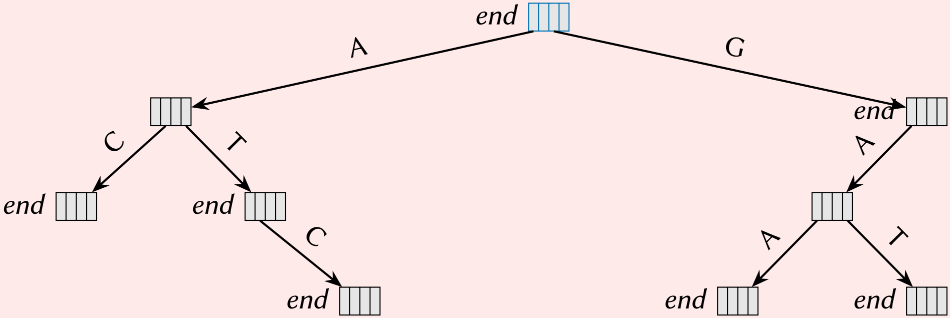

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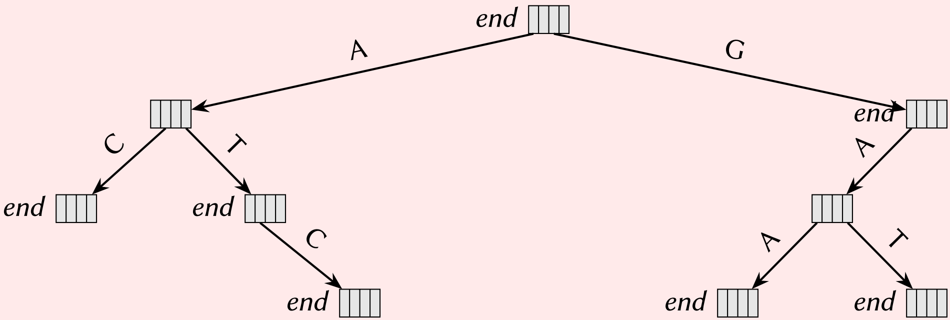

Printing all strings in-order

Perform a pre-order traversal starting at the root. For each node n:

- print the path from root to node n if $n.end$ is set;
- pre-order traverse all children in-order of alphabet symbols.

Tries: special-purpose sets and dictionaries

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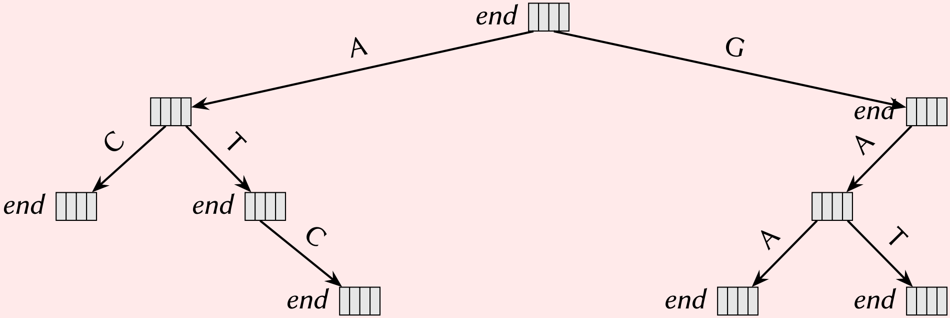

Printing all strings in-order with prefix W

- Follow a path according to the string symbols of W to node m.
- Perform a pre-order traversal starting at the node m.

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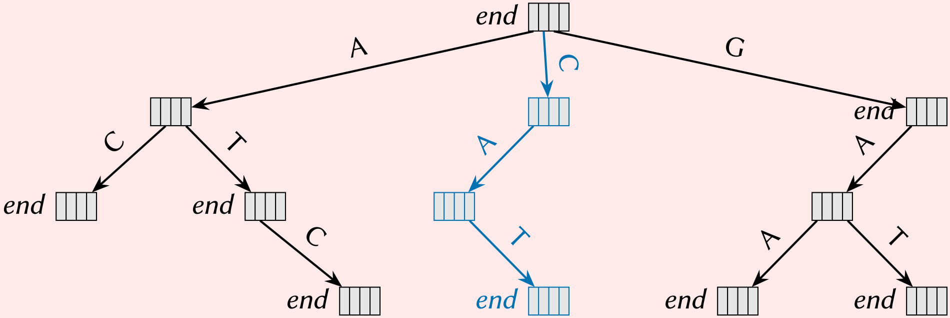

Finetuning

- To deal with big alphabets:
  use a dictionary with $\mathcal{A}$-symbols as keys at each node to store all edges.

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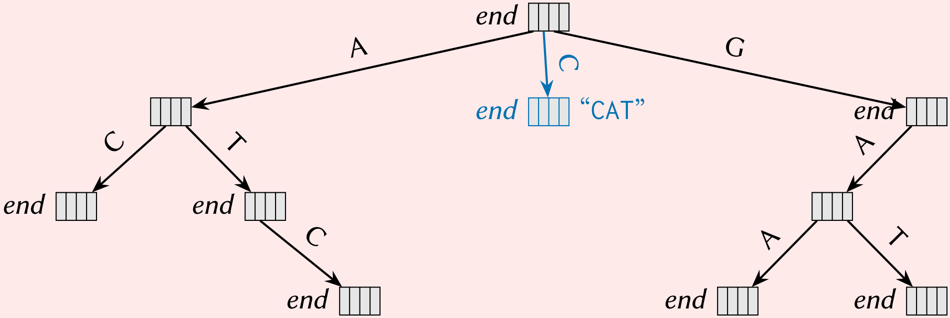


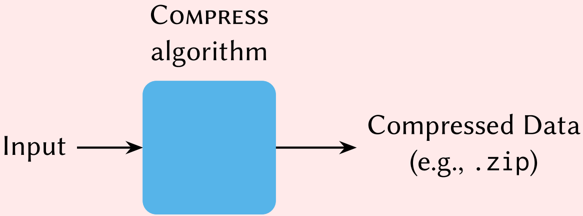
Finetuning

- To deal with big alphabets:
  use a dictionary with $\mathcal{A}$-symbols as keys at each node to store all edges.
- To *compress* non-branching paths: nodes can represent strings of symbols.

Data compression

Input

Data compression



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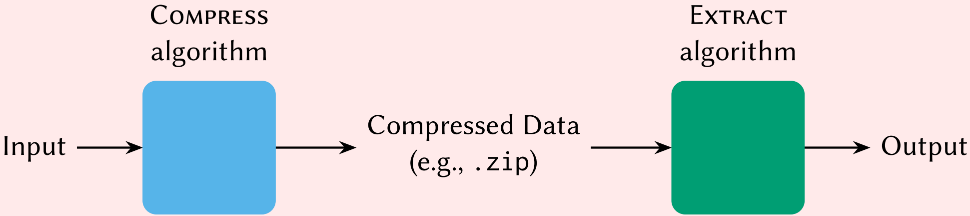

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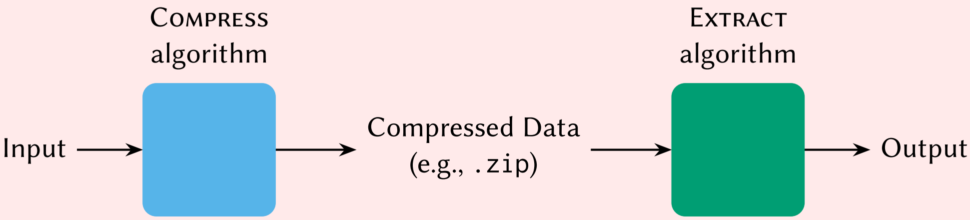

Lossless compression: The **input** must be equivalent to the **output**!

Limits of compression

Theorem

No algorithm A can compress every input I.

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Repeaded application of A on the *input* will eventually lead to zero bits!

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  E.g., we can compress at-most 2^{32} out of 2^{64} values from 8 B to 4 B.

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Conceptually: We need *structure* in the input to be able to reliably compress that input!

A simple structure: Small alphabets

Consider DNA strings over the alphabet $\mathcal{A} = \{A, C, T, G\}$.

An usual DNA string $\mathcal{S}$ represented by N characters takes up $N\text{B} = 8N\text{bit}$.

How many bytes do we need to represent $\mathcal{S}$?

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- A single byte holds 8 bit.

Hence, we need at-most $\frac{8N}{4} = 2N\text{bit}$.
A compression ratio of 0.25.

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How many bytes do we need to represent $\mathcal{S}$?

- We have $|\mathcal{A}| = 4$ distinct values.
- We can represent 4 distinct values with 2 bit: “00”, “01”, “10”, and “11”.
- A single byte holds 8 bit.

Hence, we need at-most $\frac{8N}{4} = 2N$bit.
A compression ratio of 0.25.

From *bits* to *bytes*

We can store *four DNA characters* in one byte. Can we store $\mathcal{S}$ in $\frac{2N}{8}$B using the above?

A simple structure: Small alphabets

Consider DNA strings over the alphabet $\mathcal{A} = \{A, C, T, G\}$.

An usual DNA string $\mathcal{S}$ represented by N characters takes up $N\text{B} = 8N\text{bit}$.

How many bytes do we need to represent $\mathcal{S}$?

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- We can represent 4 distinct values with 2 bit: “00”, “01”, “10”, and “11”.
- A single byte holds 8 bit.

Hence, we need at-most $\frac{8N}{4} = 2N\text{bit}$.
A compression ratio of 0.25.

From *bits* to *bytes*

We can store *four DNA characters* in one byte. Can we store $\mathcal{S}$ in $\frac{2N}{8}\text{B}$ using the above?

No! Where in the last byte would our string end?
E.g., “ACTGA” takes 10 bit (1.25 B).

A common structure: Repetition

Consider the following string of bits:

`00000000000000011111110000000011111111111000000000011`

A common structure: Repetition

Consider the following string of bits:

```
00000000000000011111110000000011111111111000000000011
```

15 zeros | 7 ones | 8 zeros | 11 ones | 10 zeros | 2 ones

A common structure: Repetition

Consider the following string of bits:

000000000000000111111100000000111111111110000000000 11

15 zeros 7 ones 8 zeros 11 ones 10 zeros 2 ones

Number	(in 4-bit binary)
15	1111
7	0111
8	1000
11	1011
10	1010
2	0010

A common structure: Repetition

Consider the following string of bits:

```
111101111100010111101000010
```

$\underbrace{\texttt{1111}}_{\texttt{0s}}\underbrace{\texttt{0111}}_{\texttt{1s}}\ldots$

0s 1s 0s 1s 0s 1s

Number	(in 4-bit binary)
15	1111
7	0111
8	1000
11	1011
10	1010
2	0010

From $15 + 7 + 8 + 11 + 10 + 2 = 53$ bit to $6 \cdot 4 = 24$ bit.

A common structure: Repetition

Consider the following string of bits:

`000000000000000001111111000000001111111111100000000001 1`

17 zeros | 7 ones | 8 zeros | 11 ones | 10 zeros | 2 ones

Number	(in 4-bit binary)
15	1111
7	0111
8	1000
11	1011
10	1010
2	0010

A common structure: Repetition

Consider the following string of bits:

`111100000001001111000101110100010`

`0s 1s 0s 1s 0s 1s 0s 1s`

Number	(in 4-bit binary)
15	`1111`
7	`0111`
8	`1000`
11	`1011`
10	`1010`
2	`0010`

From $17 + 7 + 8 + 11 + 10 + 2 = 55$ bit to $8 \cdot 4 = 32$ bit.

A common structure: Repetition

Consider the following string of bits:

`11110000001001111000101110100010`

`0s 1s 0s 1s 0s 1s 0s 1s`

Number	(in 4-bit binary)
15	1111
7	0111
8	1000
11	1011
10	1010
2	0010

Run-length encoding: *simple idea* with good results on *bitmaps*.

Another common structure: Using symbol frequencies

Consider *simple* text written in the English language.

Another common structure: Using symbol frequencies

Consider *simple* text written in the English language.

The text uses the following 66 symbols "frequently":

- Digits `0123456789`: 10 symbols.
- Lower-case letters "a"–"z": 26 symbols.
- upper-case letters "A"–"Z": 26 symbols.
- Punctuation " ", ".", ",", "!": 4 symbols.

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Even in these "frequent" symbols, some are much rarer than others: "x" versus "e".

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Even in these "frequent" symbols, some are much rarer than others: "x" versus "e".

Idea. Use fewer bits for frequent characters, more for rare characters.

Another common structure: Using symbol frequencies

Consider the string “anna can scan a can!”.

Another common structure: Using symbol frequencies

Consider the string "anna can scan a can!".

Symbol	Count	Bit pattern
'a'	6	
'n'	5	
' '	4	
'c'	3	
's'	1	
'!'	1	

The string has *6 distinct symbols*: at-least 3 bits if *all the same length*.

Another common structure: Using symbol frequencies

Consider the string "`anna can scan a can!`".

Symbol	Count	Bit pattern
'a'	6	`000`
'n'	5	`001`
' '	4	`010`
'c'	3	`011`
's'	1	`100`
'!'	1	`101`

The string has *6 distinct symbols*: at-least 3 bits if *all the same length*.

Another common structure: Using symbol frequencies

Consider the string "anna can scan a can!".

Symbol	Count	Bit pattern
'a'	6	0
'n'	5	1
' '	4	00
'c'	3	01
's'	1	10
'!'	1	11

Attempt 1. The most-frequent symbols get the shortest bit patterns.

```
a n n a    c  a n    s  c  a n    a    c  a n !
0 1 1 0 00 01 0 1 00 10 01 0 1 00 0 00 01 0 1 11
```

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a n n a    c  a n    s  c  a n    a    c  a n !
0 1 1 0 00 01 0 1 00 10 01 0 1 00 0 00 01 0 1 11
```

```
0110000101001001010000001011 1
```

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Consider the string "`anna can scan a can!`".

Symbol	Count	Bit pattern
'a'	6	`0`
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`01100001010010010100000010111` ← 29 bit *instead of at-least* 60 bit.

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a?

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$\underbrace{\texttt{01}}_{c?}$

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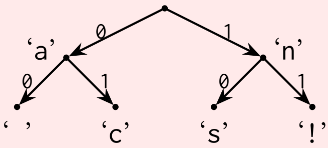
'n'	5	`1`
' '	4	`00`
'c'	3	`01`
's'	1	`10`
'!'	1	`11`



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`01100001010010010100000010111` ⟵ 29 bit *instead of at-least* 60 bit.

Issue. The bit pattern of one symbol (e.g., a) is a *prefix* of other symbols!

Another common structure: Using symbol frequencies

Consider the string "`anna can scan a can!`".

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'a'	6	`0`
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' '	4	`110`
'c'	3	`1110`
's'	1	`11110`
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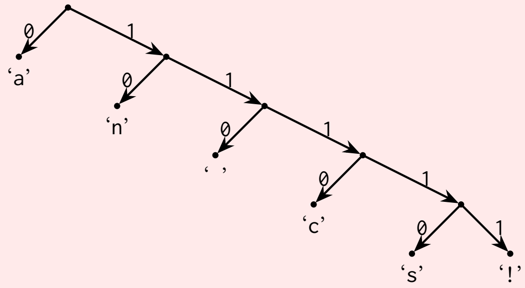
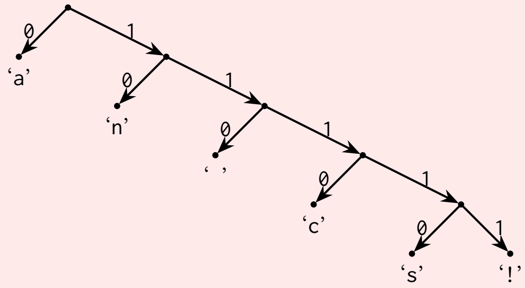


Attempt 2. The most-frequent symbols get the shortest *prefix-free* bit patterns.

Another common structure: Using symbol frequencies

Consider the string "anna can scan a can!".

Symbol	Count	Bit pattern
'a'	6	0
'n'	5	10
' '	4	110
'c'	3	1110
's'	1	11110
'!'	1	11111



Attempt 2. The most-frequent symbols get the shortest *prefix-free* bit patterns.

```
a n  n  a     c    a n      s     c    a n      a     c    a n  !
0 10 10 0 110 1110 0 10 110 11110 1110 0 10 110 0 110 1110 0 10 11111
```

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Consider the string “`anna can scan a can!`”.

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‘a’	6	`0`
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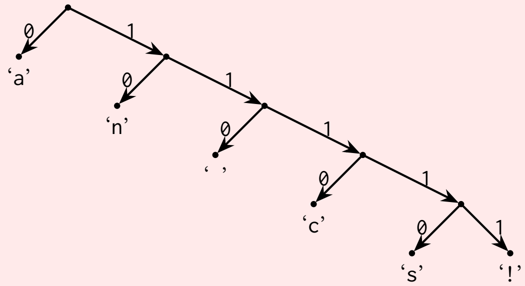
‘ ’	4	`110`
‘c’	3	`1110`
‘s’	1	`11110`
‘!’	1	`11111`



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```
a n  n  a     c    a n      s     c    a n     a     c    a n  !
0 10 10 0 110 1110 0 10 110 11110 1110 0 10 110 0 110 1110 0 10 11111
```

`01010011011100101101111011100101100110111001011111` ← 50 bit.

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Consider the string “anna can scan a can!”.

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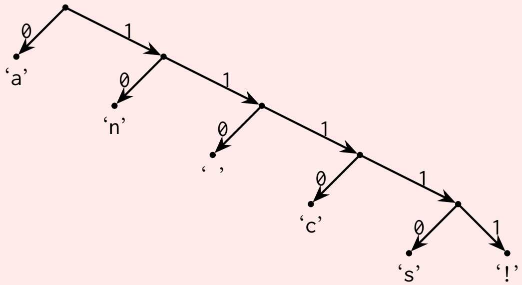

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⋎

a

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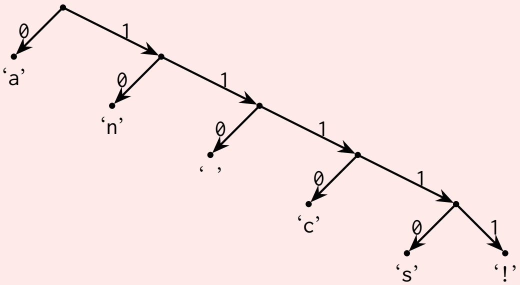

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`01010011011100101101111011100101100110111001011111` ← 50 bit.

`a n n a ‘ ’ c a n ‘ ’ s c a n ‘ ’ a ‘ ’ c a n ‘!’`

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‘a’	6	0
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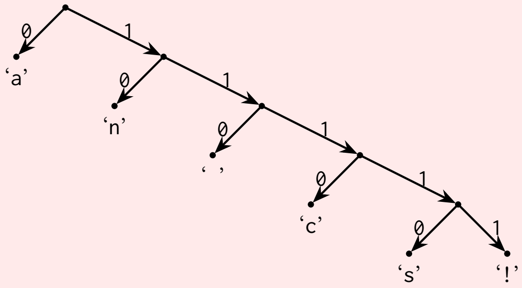

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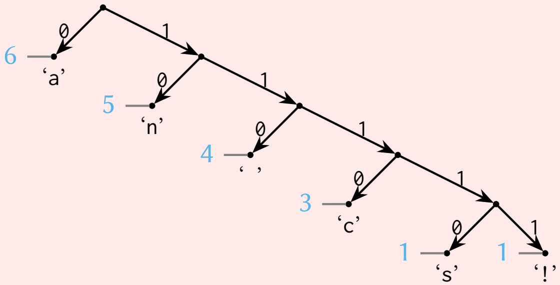

Questions. How to construct the bit patterns and are these patterns optimal?

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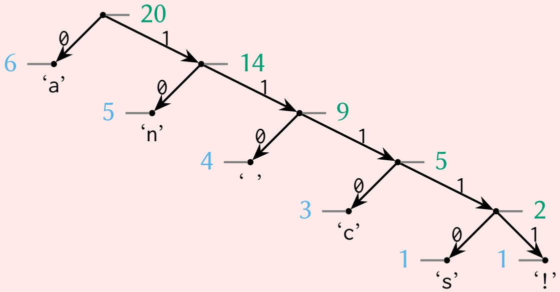


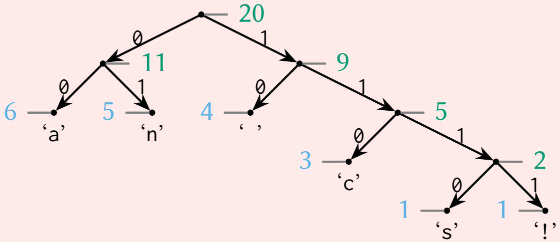
Questions. How to construct the bit patterns and are these patterns optimal?

$$6 \cdot 1 + 5 \cdot 2 + 4 \cdot 3 + 3 \cdot 4 + 1 \cdot 5 + 1 \cdot 5 = 50.$$

Another common structure: Using symbol frequencies

Consider the string “anna can scan a can!”.

Symbol	Count	Bit pattern
‘a’	6	00
‘n’	5	01
‘ ’	4	10
‘c’	3	110
‘s’	1	1110
‘!’	1	1111



Questions. How to construct the bit patterns and are these patterns optimal?

$$6 \cdot 2 + 5 \cdot 2 + 4 \cdot 2 + 3 \cdot 3 + 1 \cdot 4 + 1 \cdot 4 = 47.$$

Huffman coding

Problem

Given an alphabet $\mathcal{A}$ and symbol-frequencies $f : \mathcal{A} \to [0, 1]$,
Produce *prefix-free bit patterns* for all symbols in $\mathcal{A}$ such that these patterns are optimal.

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Optimal No other bit patterns will compress strings $\mathcal{S}$ over $\mathcal{A}$ more
Assuming symbols counts in $\mathcal{S}$ agree with f.

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Assuming symbols counts in $\mathcal{S}$ agree with f.

Algorithm HuffmanPFTrie(f):

$Q :=$ a min-priority queue.
for all $\sigma \in \mathcal{A}$ **do**
  Make a leaf-node n labeled σ.
  Add $(n, f(\sigma))$ to Q with priority $f(\sigma)$.
while $|Q| \geq 2$ **do**
  $(n_0, p_0) :=$ DelMin(Q), $(n_1, p_1) :=$ DelMin(Q).
  Create a node n with children n_0 labeled `0`, n_1 labeled `1`.
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return n with $(n, p) :=$ DelMin(Q).

Huffman coding

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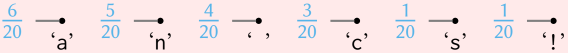

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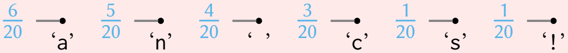

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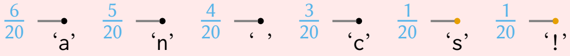

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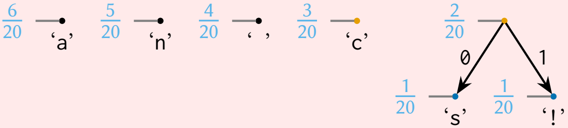

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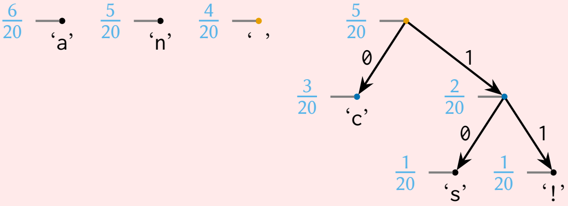

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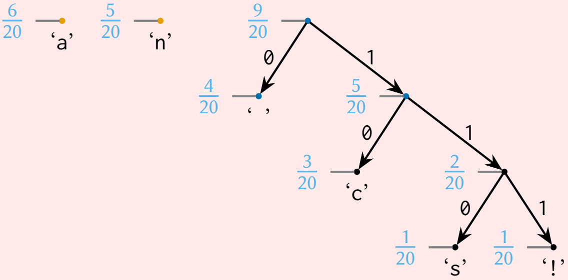

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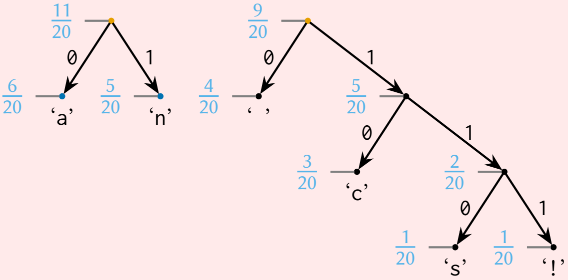

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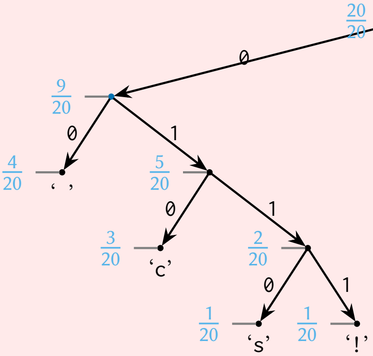

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return n with $(n, p) :=$ DELMIN(Q).

Huffman coding

Symbol	Count	Frequency
‘a’	6	$\frac{6}{20}$
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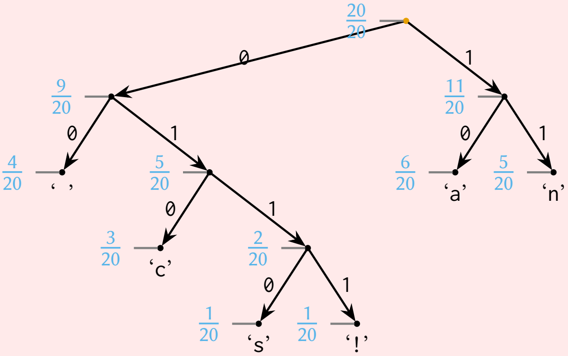

Algorithm HUFFMANPFTRIE(f):

…
while $|Q| \geq 2$ **do**
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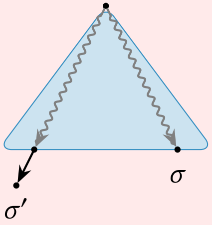
Property 1

Let $\sigma \in \mathcal{A}$ be the symbol with lowest frequency f.
Any optimal prefix-free trie for $\mathcal{A}, f$ can be changed such that the path to σ is longest.



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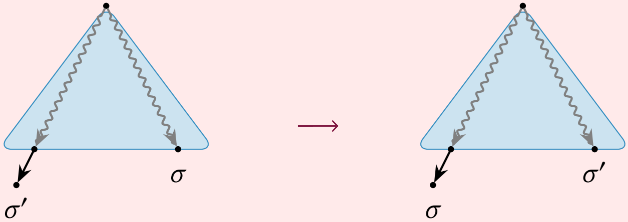

Huffman coding

Property 2

Let $\sigma_1, \sigma_2 \in \mathcal{A}$ be the symbols with lowest frequency f.
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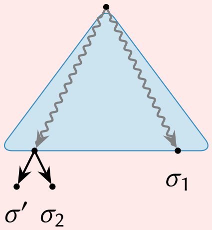

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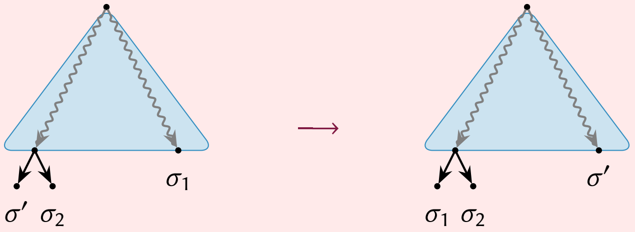


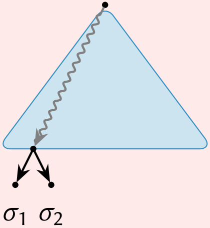
Huffman coding

Property 3

Let $\sigma_1, \sigma_2 \in \mathcal{A}$ be symbols represented by children n_0, n_1 of node n in trie T.
Let T' be the prefix-free trie for $\mathcal{A}', f'$ with

- $\mathcal{A}' = \mathcal{A} \setminus \{\sigma_2\}$;
- $f' = \{\sigma \mapsto f(\sigma) \mid \sigma \in \mathcal{A} \setminus \{\sigma_1, \sigma_2\}\} \cup \{\sigma_1 \mapsto f(\sigma_1) + f(\sigma_2)\}$; and
- leafs n_0, n_1 removed from n and n made to represent σ_1.

The trie T is optimal for $\mathcal{A}, f$ if and only if T' is optimal for $\mathcal{A}', f'$.



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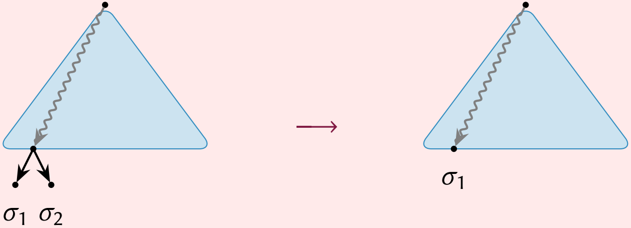

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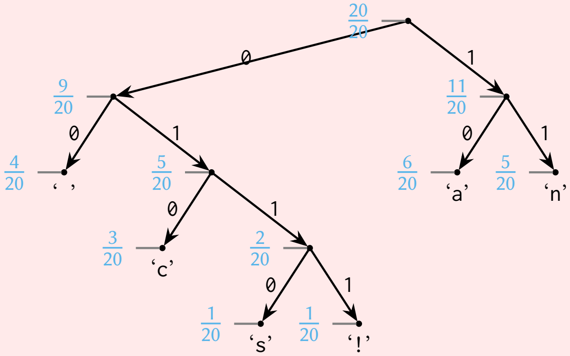

Theorem

The HuffmanPFTrie *algorithm builds an optimal prefix-free code.*

Proof (sketch)

HuffmanPFTrie follows Property 1–3.

Beyond Huffman: Frequent strings

Huffman looks at frequent symbols from an alphabet.

- We can generalize these ideas to *frequent* sequences of symbols.
- Tries can be used to efficiently manage *frequency* data for substrings in an input.
- Challenge: which substrings to consider?
  E.g., fixed length, maximum length, ...

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Many variations of this idea used in practice, e.g., `.zip`, `.gif`, ...

Basic substring search: Searching a needle in a haystack

Problem

Given strings $\mathcal{S}$ (the haystack) and P (the needle or pattern), return the first position in $\mathcal{S}$ at which P occurs (if any).

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Searching P = "*example*"

```
“an example of words”
 example
```

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Complexity

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Algorithm BasicStringSearch($\mathcal{S}$, P):

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} $\Theta(|P|)$

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$\Theta((|\mathcal{S}| - |P|)|P|) = \Theta(|\mathcal{S}||P|)$

Complexity

Substring search: Room for improvement

Searching $P =$ “*string*”

“a strong string”

Substring search: Room for improvement

Searching P = "*string*"

```
“a  strong  string”
 string
```

Substring search: Room for improvement

Searching $P =$ "*string*"

```
“a strong string”
  string
```

Substring search: Room for improvement

Searching $P =$ "*string*"

```
" a   s t r o n g   s t r i n g "
      s t r i n g
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s t r i n g

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Searching $P =$ "*string*"

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" a   s t r o n g   s t r i n g "
            s t r i n g
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Substring search: Room for improvement

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Substring search: Room for improvement

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“a strong string”
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Substring search: Room for improvement

Searching P = "$ACACGT$"

" A C A C A C A C G T "

Substring search: Room for improvement

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" A C A C A C A C G T "

A C A C G T

Substring search: Room for improvement

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A C A C G T

Substring search: Room for improvement

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A C A C G T

Substring search: Room for improvement

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```
" A C A C A C A C G T "
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Substring search: Room for improvement

Searching $P =$ "$ACACGT$"

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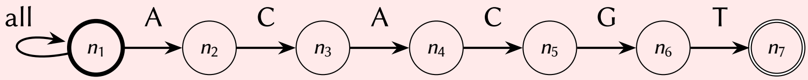
A C A C G T

Encode search patterns as an automaton

Finite automata

A finite automaton is a *graph* with

- a single *initial* node;
- zero-or-more *final* nodes;
- edges labeled with *symbols*.



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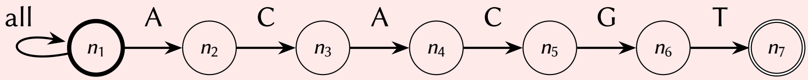

Typically, we refer to nodes as *states* and edges as *transitions*.

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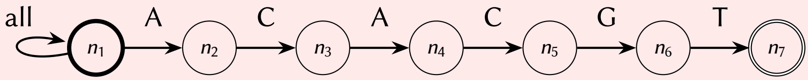

Running a finite automaton

We can use a string as *input* to the automaton to decide which path to follow.

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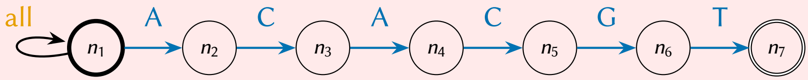

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"ACACACACGT".

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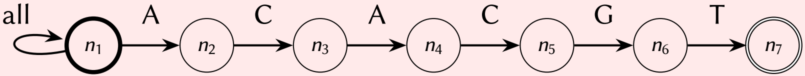

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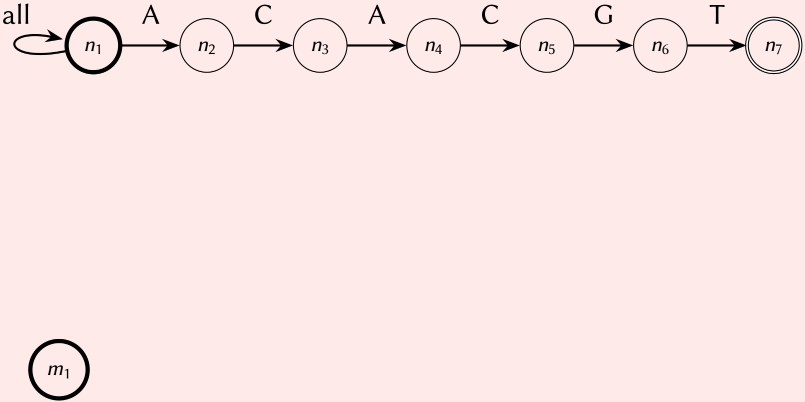
For efficiency: we want a *deterministic* automaton: an automaton without choices!

Encode search patterns as an automaton



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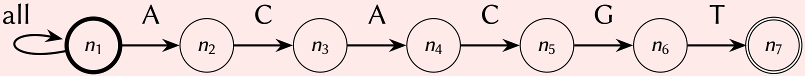

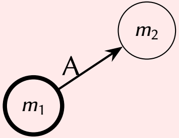

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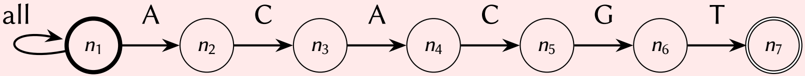

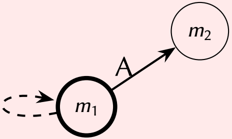

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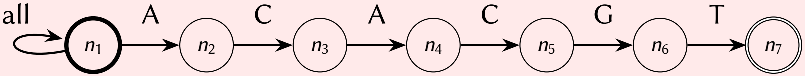

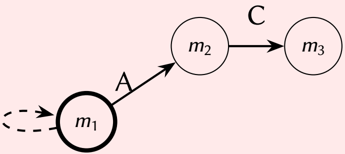

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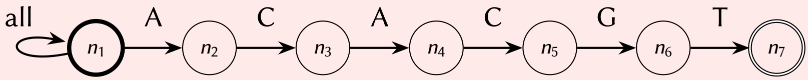

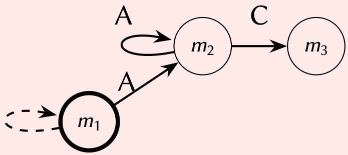

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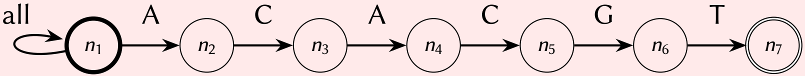

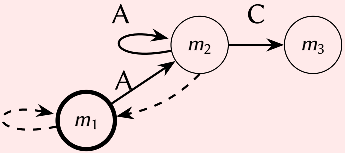

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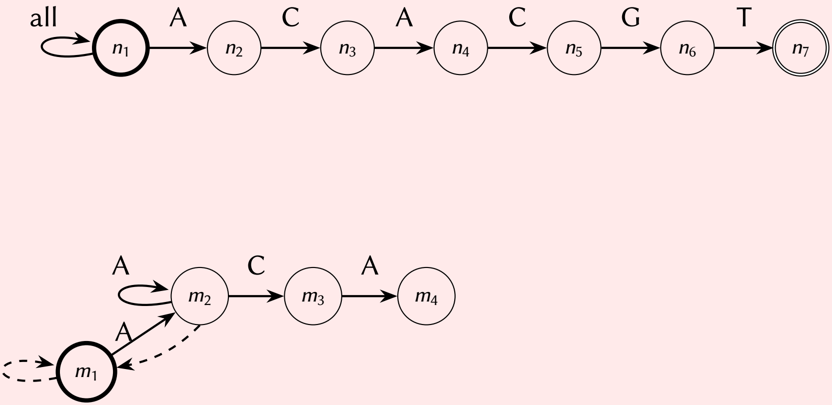

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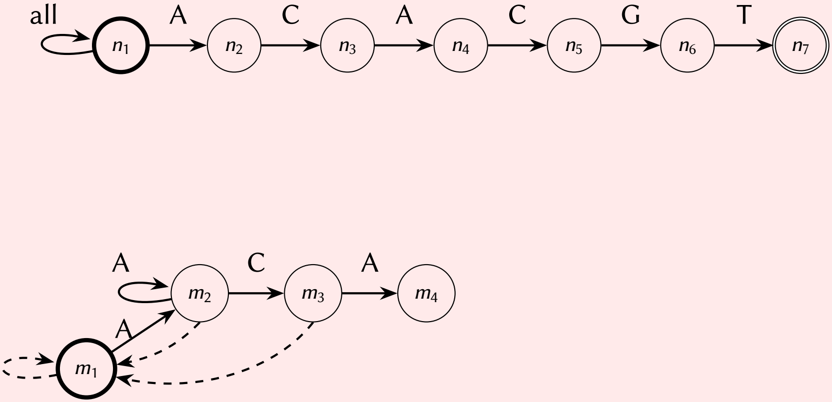

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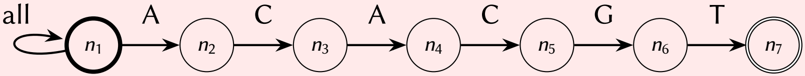

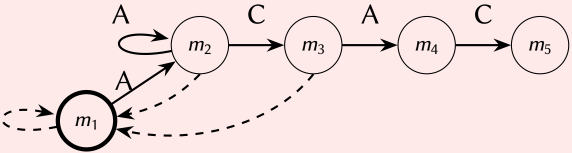

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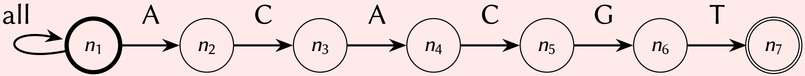

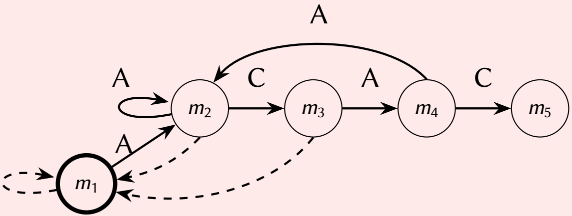

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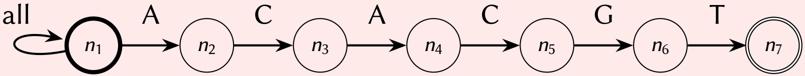

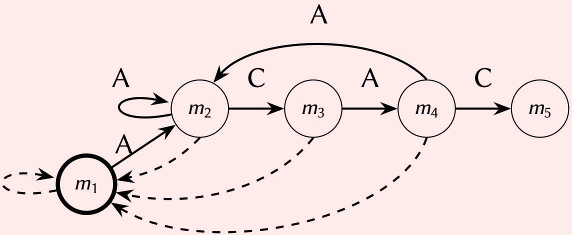

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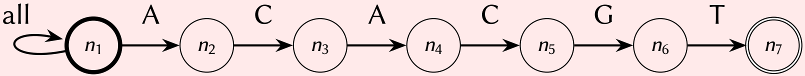

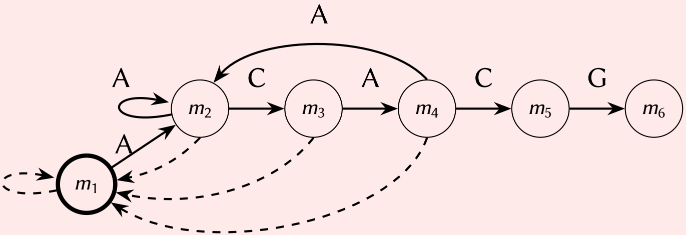

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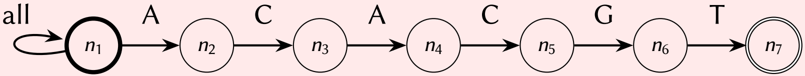

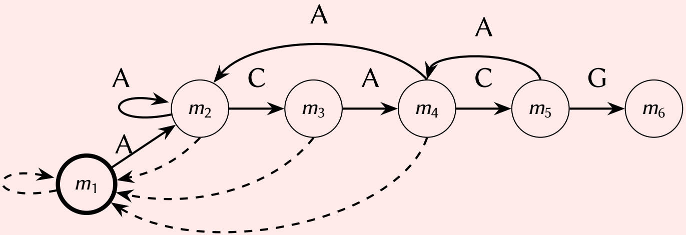

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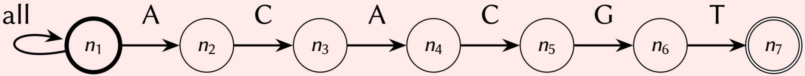

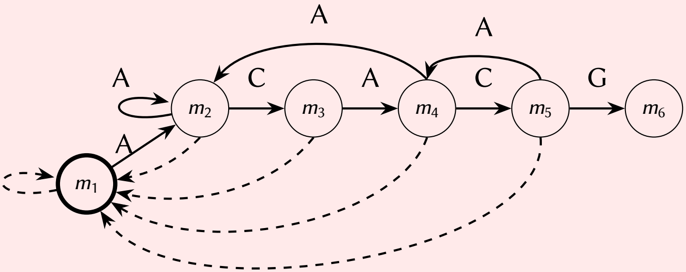

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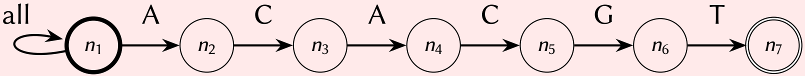

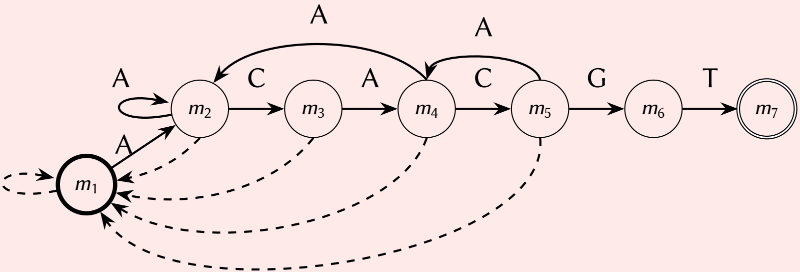

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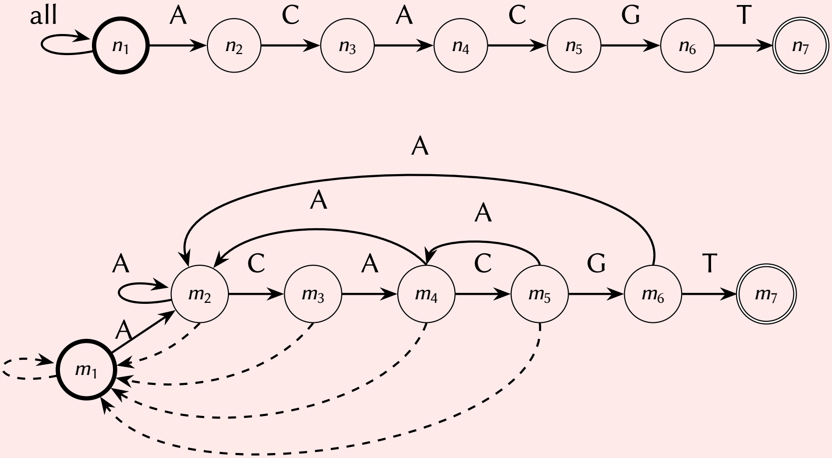

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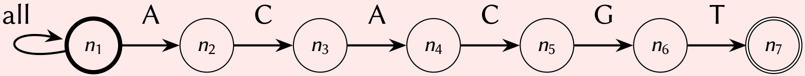

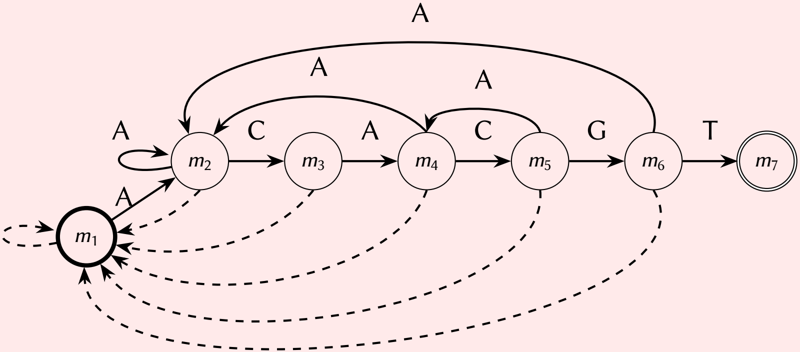

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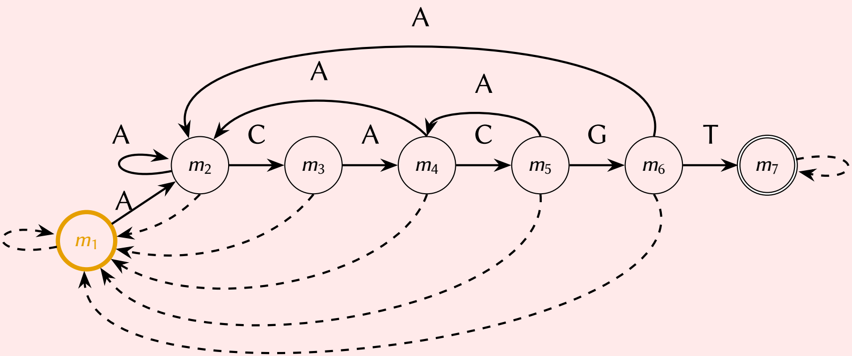

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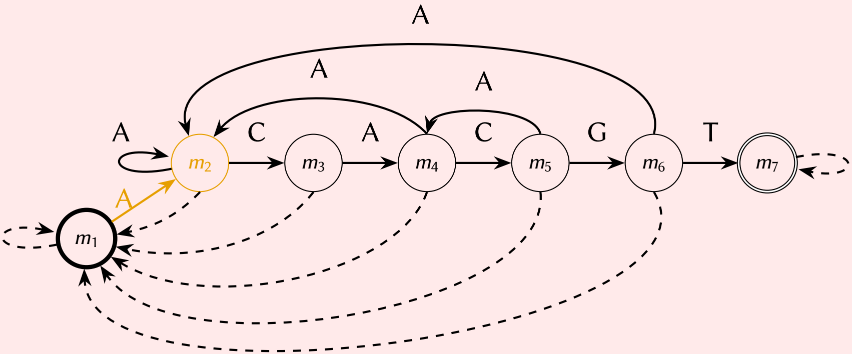

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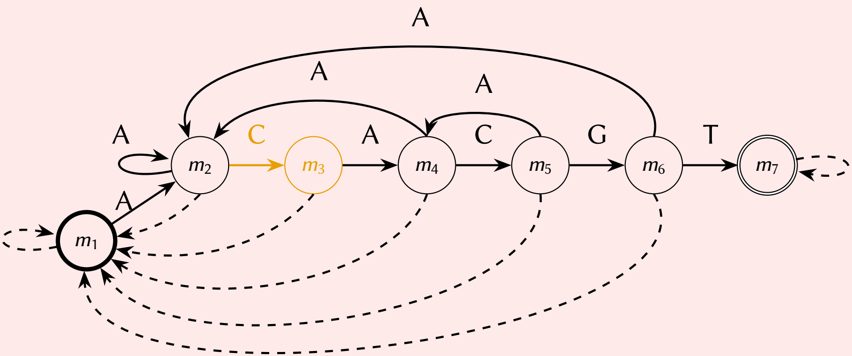

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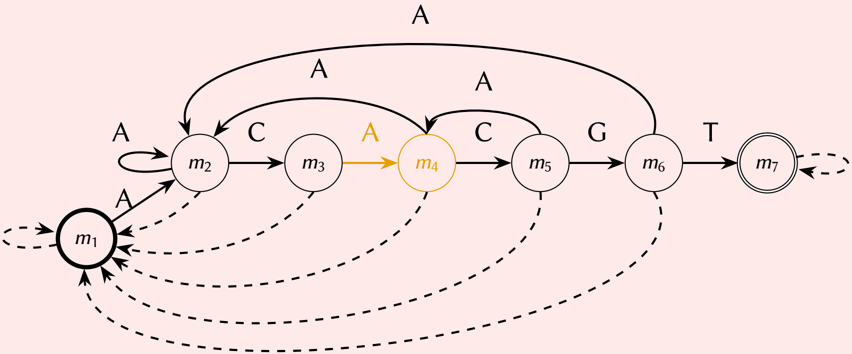

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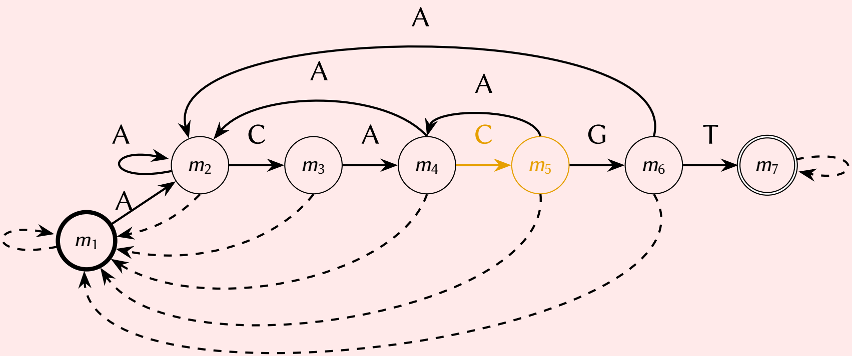

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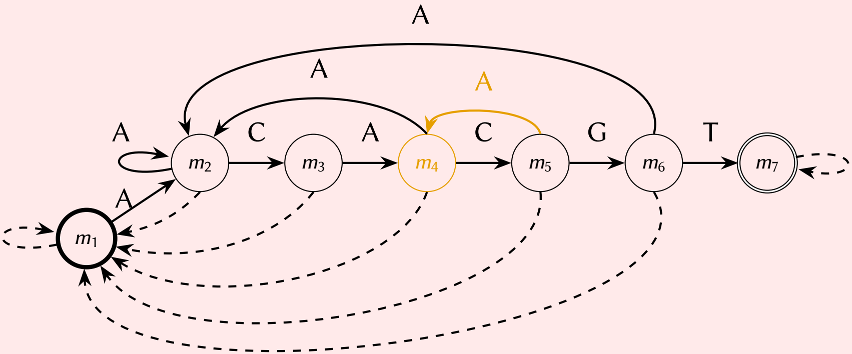

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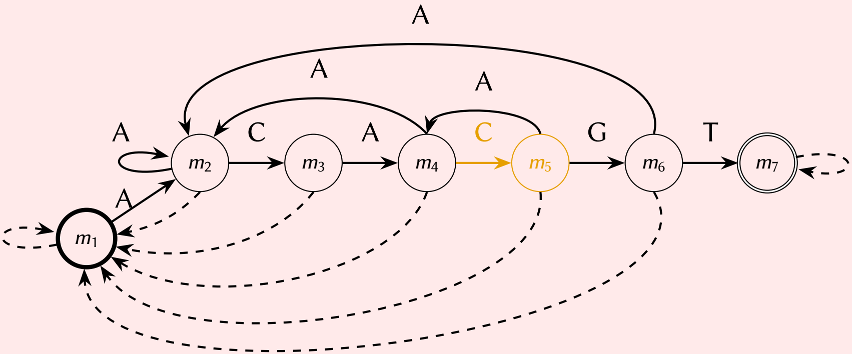

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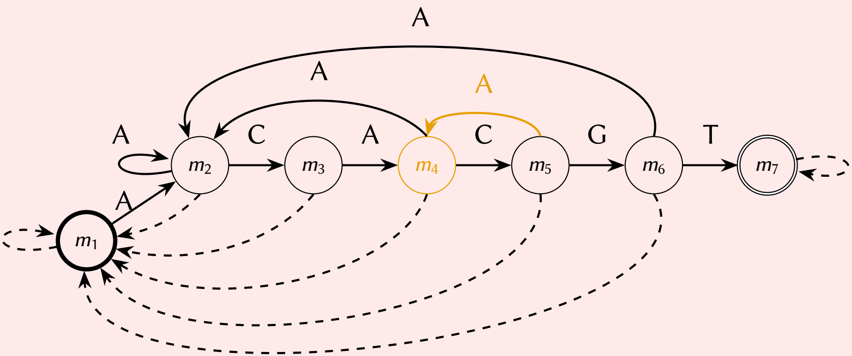

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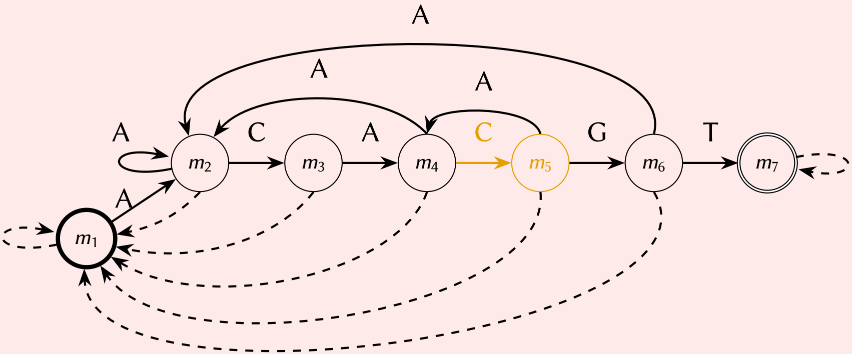

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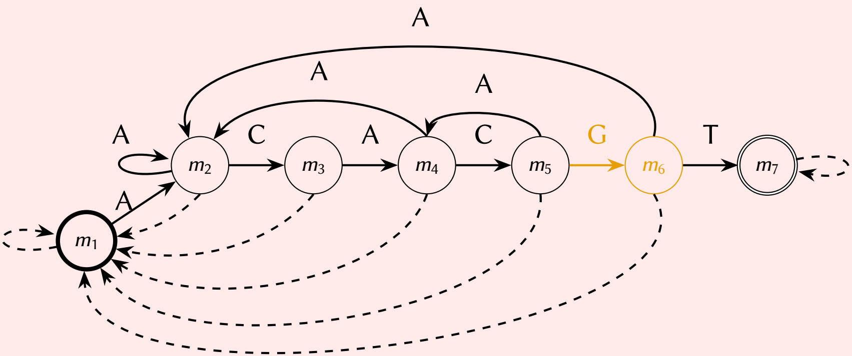

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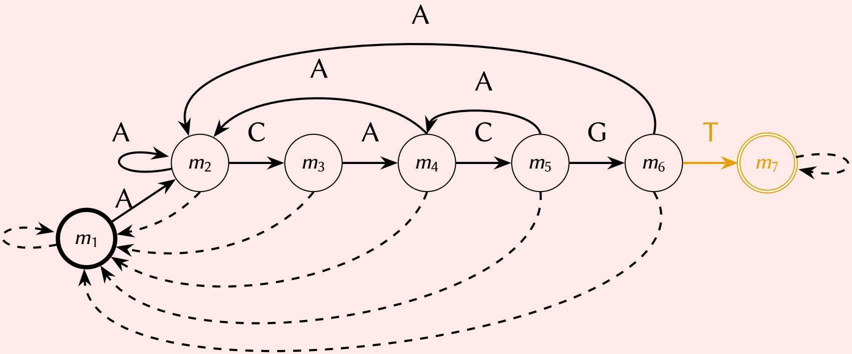

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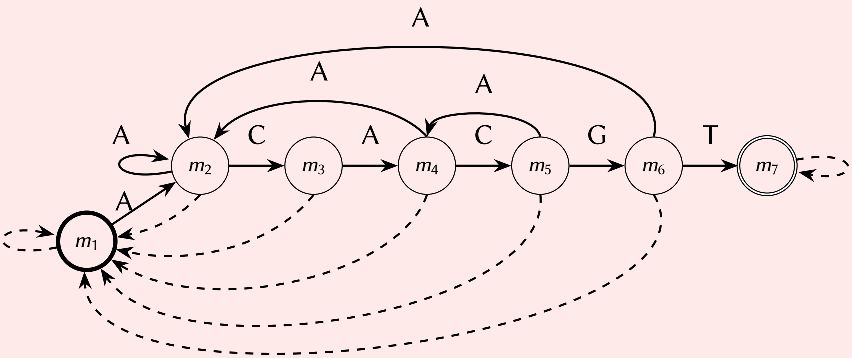

Complexity of running a *deterministic* finite automaton with input $\mathcal{S}$

- Always in exactly *one* state.
- We perform at-most $\Theta(|\mathcal{S}|)$ *state transitions* in the automaton.
- Need efficient representation of the *transitions* (per state): e.g., hash table.

More general search patterns: Regular expressions

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- (e) desribes R.
- $e_1 e_2$ describes $\{\textsc{Concatenate}(\mathcal{S}_1, \mathcal{S}_2) \mid \mathcal{S}_1 \in R_1 \wedge \mathcal{S}_2 \in R_2\}$.
- $e_1 \mid e_2$ describes $R_1 \cup R_2$.
- e^* describes any sequence of strings in R.

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See SFWRENG 2FA3: Discrete Mathematics with Applications II.

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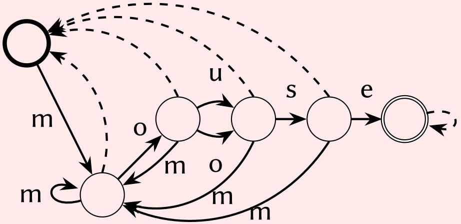

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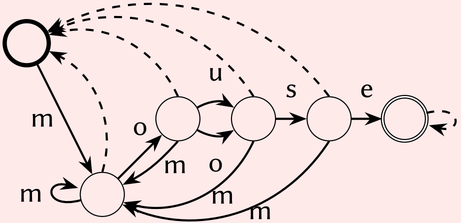

Deterministic finite automata can grow *very* large

Alternatively, you can run a *nondeterminstic* finite automaton: an automaton with choices!

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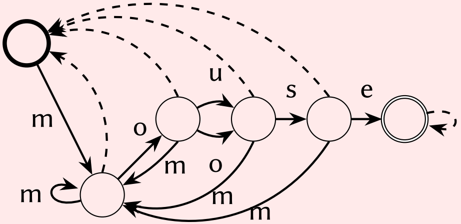

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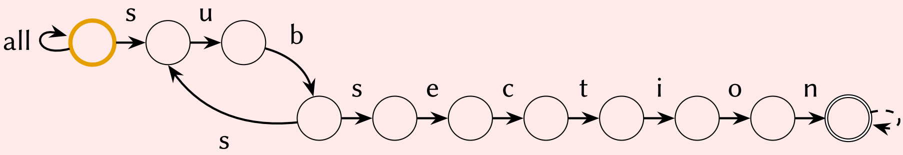

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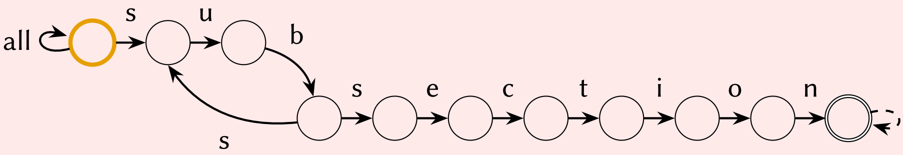

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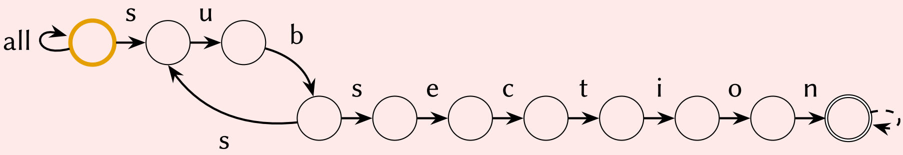

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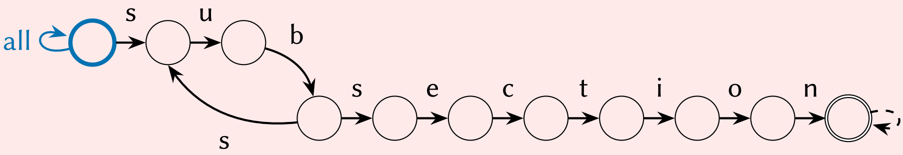

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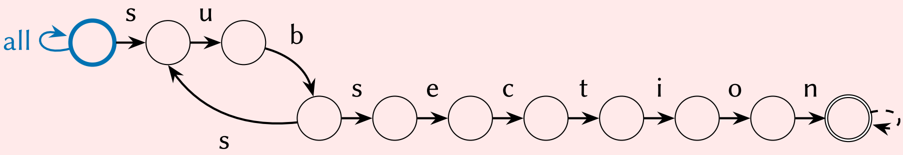

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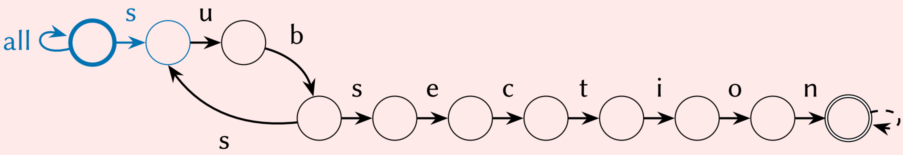

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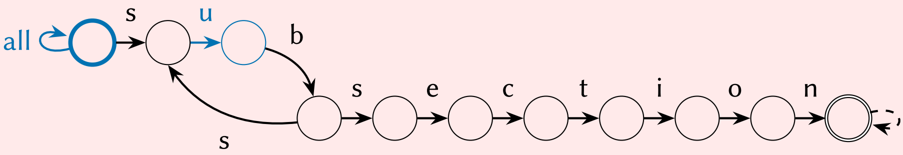

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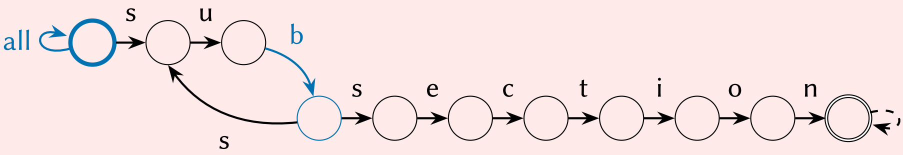

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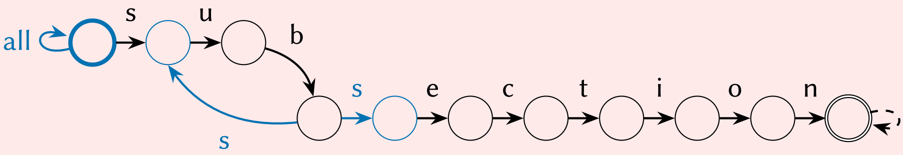

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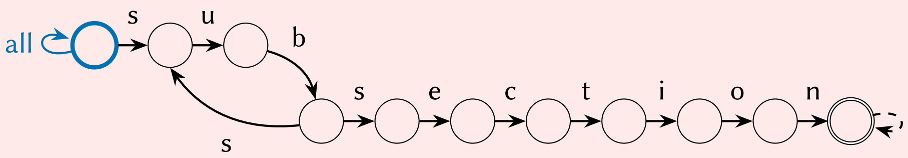

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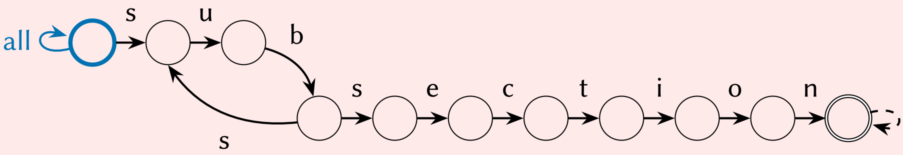

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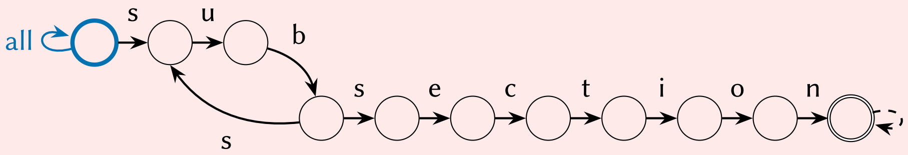

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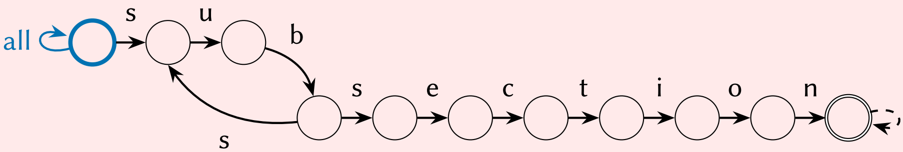

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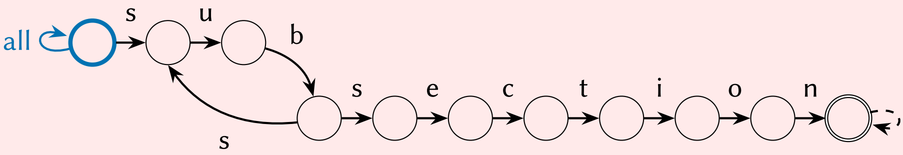

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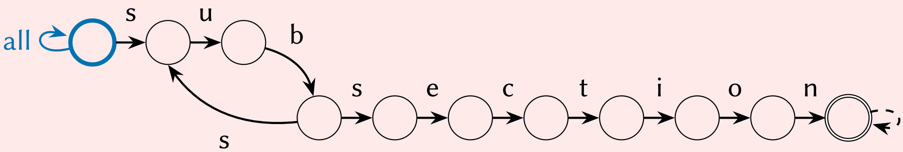

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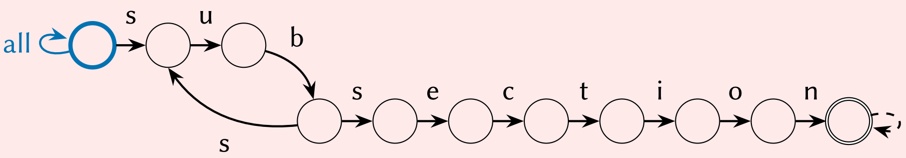

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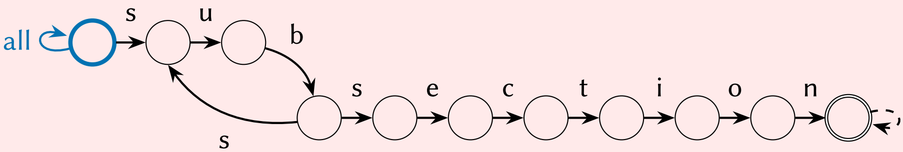

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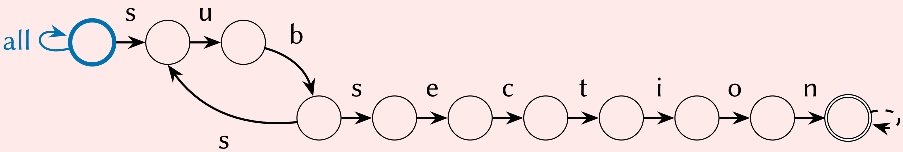

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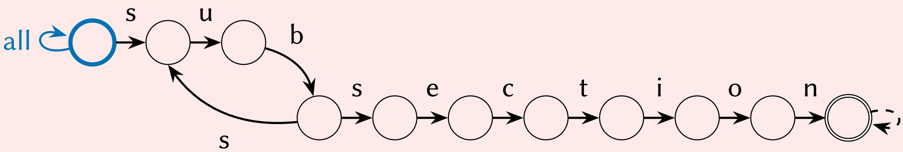

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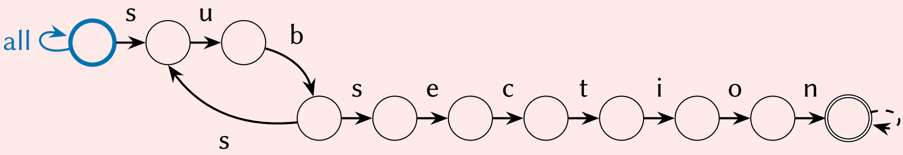

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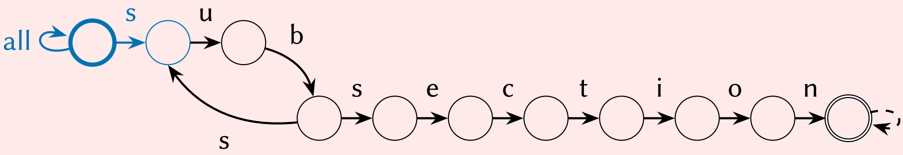

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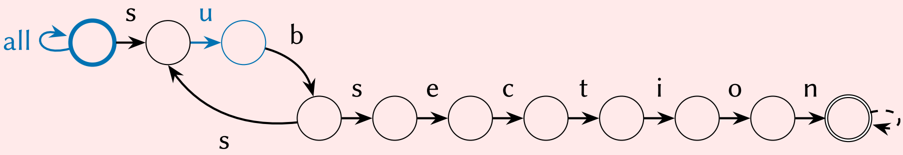

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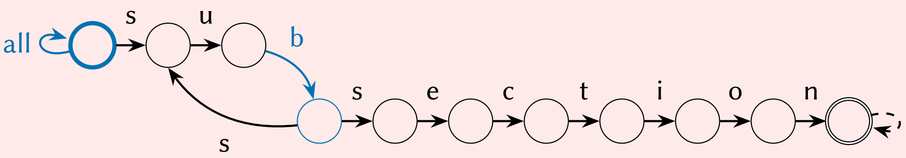

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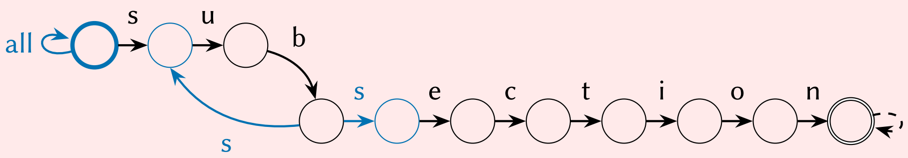

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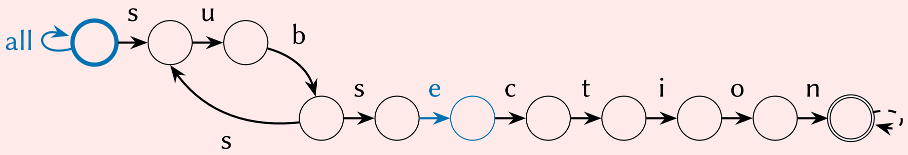

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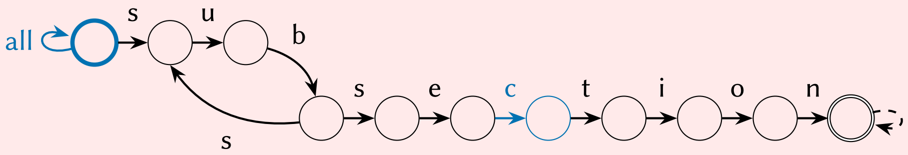

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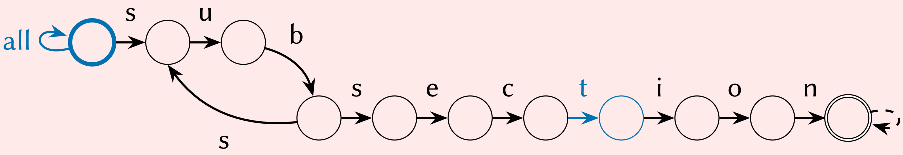

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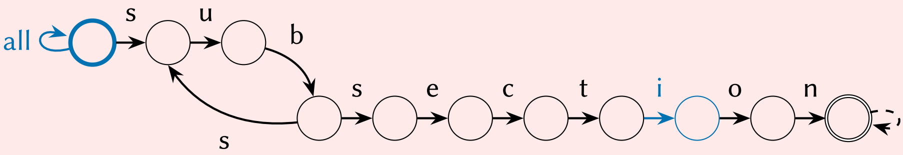

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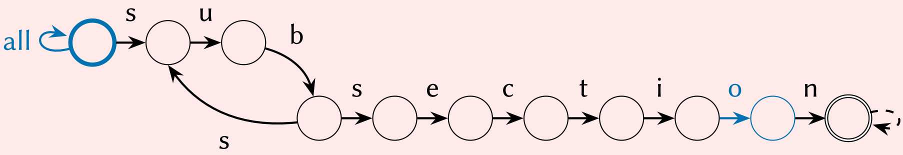

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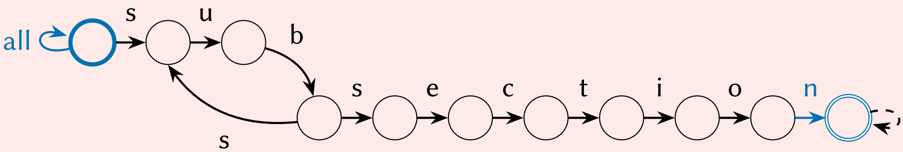

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- Automata-based searching in string can be very fast: core component in lexers, parsers, and compilers.
- Many *RegExp* libraries are *regular expression like*: they support non-regular features. Consequently, many such libraries use *shamefully slow* backtracking algorithms: Worst-case exponential complexity, even when searching for *simple patterns*.

Substring search in less-than $\Theta(|\mathcal{S}|)$ steps

Searching $P =$ "*great*"

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“a n   e x a m p l e   o f   w o r d s”
 g r e a t
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How can I skip checking most letters in $\mathcal{S}$?

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  When it does not matches: another opportunity to jump!

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With some preprocessing on P one can precompute how to jump around optimally: the Boyer-Moore algorithm.