Set theory

Complement in Σ^*:

$$\overline{L} = \Sigma^* - L$$

associative:

$$(A \cup B) \cup C = A \cup (B \cup C),$$
$$(A \cap B) \cap C = A \cap (B \cap C),$$
$$(AB)C = A(BC).$$

commutative:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

null set

null set $\emptyset$ is the identity for $\cup$ and annihilator for set concatenation

$A \cup \emptyset = A$ and $A\emptyset = \emptyset A = \emptyset$

set $\{\epsilon\}$ is an identity for set concatenation $\{\epsilon\}A = A\{\epsilon\} = A$

Set union and intersection are distributive over set concatenation

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Set concatenation distributes over union

$$A(B \cup C) = AB \cup AC$$
$$(A \cup B)C = AC \cup BC$$

regularity

Important

$$\hat{\delta}(q, \epsilon) = q$$
$$\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$$

Important

a subset $A \subset \Sigma^*$ is regular if and only if there exists a DFA M such that $\mathcal{L}(M) = L$

Important

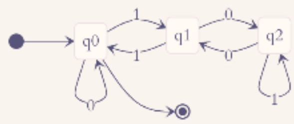
All finite languages are regular, but not all regular languages are finite

examples

Show L is regular where $L = \{x \mid x\%3 = 0 \cup x = \epsilon\}$, with $\Sigma = \{0, 1\}$

Three states, q_0, q_1, q_2, where q_0 denotes the string mod 3 is 0, q_1 denotes the string mod 3 is 1, and q_2 denotes the string mod 3 is 2.

$\forall x \in \{0,1\} \to \delta(q_0, x) = 0 \iff \#x = 0 \; mod \; 3$, $\delta(q_0, x) = q_1 \iff \#x = 1 \; mod \; 3$, $\delta(q_0, x) = q_2 \iff \#x = 2 \; mod \; 3$



product construction

Assume that A, B are regular, there are automata

$$M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1) \quad M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$$

Thus

$$M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$$

where $Q_3 = Q_1 \times Q_2$, $s_3 = (s_1, s_2)$, $F_3 = F_1 \times F_2$, and $\delta_3((p,q), x) = (\delta_1(p,x), \delta_2(q,x))$

with $L(M_1) = A$ and $L(M_2) = B$, then $A \cap B$ is regular.

Lemma 4.1

$$\delta_3((p,q), x) = (\delta_1(p,x), \delta_2(q,x)) \; \forall x \in \Sigma^*$$

Complement set: $Q - F \in Q$

Trivial machine $\mathcal{L}(M_1) = \{\}$, $\mathcal{L}(M_2) = \Sigma^*$, $\mathcal{L}(M_3) = \{\epsilon\}$

De Morgan laws

$$A \cup B = \overline{\overline{A} \cap \overline{B}}$$

Theorem 4.2

$$L(M_3) = L(M_1) \cap L(M_2)$$

> $\overline{L}$ is regular

> $L_1 \cap L_2$ is regular

> $L_1 \cup L_2$ is regular

Define

$$M/\approx \; = (Q', \Sigma, \delta', [s], F')$$

where (13.1)

$$Q' = Q/\approx$$
$$\delta'([p], a) = [\delta(p, a)]$$
$$s' = [s]$$
$$F' = \{[p] \mid p \in F\}$$

Lemma 13.5

If $p \approx q$, then $\delta(p, a) \approx \delta(q, a)$
equivalently, if $[p] = [q]$, then $[\delta(p,a)] = [\delta(q,a)]$

Lemma 13.6

$p \in F \iff [p] \in F'$

Lemma 13.7

$$\forall x \in \Sigma^*, \hat{\delta}'([p], x) = [\hat{\delta}(p, x)]$$

Theorem 13.8

$L(M/\approx) = L(M)$

algorithm

1. Table of all pairs $\{p, q\}$
2. Mark all pairs $\{p, q\}$ if $p \in F \wedge q \notin F \vee q \in F \wedge p \notin F$
3. If there exists unmarked pair $\{p, q\}$, such that $\{\delta(p,a), \delta(q,a)\}$ is marked, then mark $\{p, q\}$
4. $p \approx q \iff \{p, q\}$ is not marked

transition function

$\hat{\Delta} : P(Q) \times \Sigma^* \to P(Q)$

$$\hat{\Delta}(A, a) = \bigcup_{p \in \hat{\Delta}(A,\epsilon)} \Delta(p, a)$$

$$= \bigcup_{p \in A} \Delta(p, a).$$

subset construction

acceptance

N accepts $x \in \Sigma^*$ if

$$\hat{\Delta}(s, x) \cap F \neq \emptyset$$

Define $L(N) = \{x \in \Sigma^* \mid N \text{ accepts } x\}$

Theorem 4.3

Every DFA $(Q, \Sigma, \delta, s, F)$ is equivalent to an NFA $(Q, \Sigma, \Delta, \{s\}, F)$ where $\Delta(p, a) = \{\delta(p, a$

Lemma 6.1

For any $x, y \in \Sigma^* \wedge A \subseteq Q$,

$$\hat{\Delta}(s, xy) = \hat{\Delta}(\hat{\Delta}(s, x), y)$$

Lemma 6.2

$\hat{\Delta}$ commutes with set union:

$$\hat{\Delta}(\bigcup_i A_i, x) = \bigcup_i \hat{\Delta}(A_i, x)$$

Let $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$ be arbitrary NFA. Let M be DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ wher

$$\begin{aligned} Q_M &= P(Q_N) \\ \delta_M(A, a) &= \hat{\Delta}_N(A, a) \\ s_M &= S_N \\ F_M &= \{A \in Q_N \mid A \cap F_N \neq \emptyset\} \end{aligned}$$

Lemma 6.3

For any $A \subseteq Q_N \wedge x \in \Sigma^*$

$$\hat{\delta}_M(A, x) = \hat{\Delta}_N(A, x)$$

Theorem 6.4

The automata M and N accept the same sets.

atomic patterns are:

- $L(a) = \{a\}$
- $L(\epsilon) = \{\epsilon\}$
- $L(\emptyset) = \emptyset$
- $L(\#) = \Sigma$: matched by any symbols
- $L(@) = \Sigma^*$: matched by any string

compound patterns are formed by combining binary operators and unary operators.

redundancy

$a^+ \equiv aa^*$, $\alpha \cap \beta = \overline{\overline{\alpha} + \overline{\beta}}$

> if α and β are patterns, then so are $\alpha + \beta, \alpha \cap \beta, \alpha^*, \alpha^+, \overline{\alpha}, \alpha\beta$

The following holds for x matches:

$L(\alpha + \beta) = L(\alpha) \cup L(\beta)$

$L(\alpha \cap \beta) = L(\alpha) \cap L(\beta)$

$L(\alpha\beta) = L(\alpha)L(\beta) = \{yz \mid y \in L(\alpha) \wedge z \in L(\beta)\}$

$L(\alpha^*) = L(\alpha)^0 \cup L(\alpha)^1 \cup \cdots = L(\alpha)^*$

$L(\alpha^+) = L(\alpha)^+$

Theorem 7.1

$\Sigma^* = L(\#^*) = L(@)$

Singleton set $\{x\} = L(x)$

Finite set: $\{x_1, x_2, \ldots, x_m\} = L(x_1 + x_2 + \cdots + x_m)$

Theorem 9

$$\begin{array}{rcll}
\alpha + (\beta + \gamma) & \equiv & (\alpha + \beta) + \gamma & (9.1) \\
\alpha + \beta & \equiv & \beta + \alpha & (9.2) \\
\alpha + \phi & \equiv & \alpha & (9.3) \\
\alpha + \alpha & \equiv & \alpha & (9.4) \\
\alpha(\beta\gamma) & \equiv & (\alpha\beta)\gamma & (9.5) \\
\epsilon\alpha & \equiv & \alpha\epsilon \equiv \alpha & (9.6) \\
\alpha(\beta + \gamma) & \equiv & \alpha\beta + \alpha\gamma & (9.7) \\
(\alpha + \beta)\gamma & \equiv & \alpha\gamma + \beta\gamma & (9.8) \\
\phi\alpha & \equiv & \alpha\phi \equiv \phi & (9.9) \\
\epsilon + \alpha^* & \equiv & \alpha^* & (9.10) \\
\epsilon + \alpha^* & \equiv & \alpha^* & (9.11) \\
\beta + \alpha\gamma \leq \gamma & \Rightarrow & \alpha^*\beta \leq \gamma & (9.12) \\
\beta + \gamma\alpha \leq \gamma & \Rightarrow & \beta\alpha^* \leq \gamma & (9.13) \\
(\alpha\beta)^* & \equiv & \alpha(\beta\alpha)^* & (9.14) \\
(\alpha^*\beta)^*\alpha^* & \equiv & (\alpha + \beta)^* & (9.15) \\
\alpha^*(\beta\alpha^*)^* & \equiv & (\alpha + \beta)^* & (9.16) \\
(\epsilon + \alpha)^* & \equiv & \alpha^* & (9.17) \\
\alpha\alpha^* & \equiv & \alpha^*\alpha & (9.18)
\end{array}$$