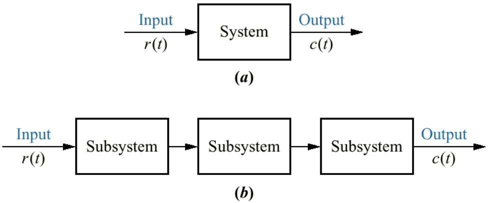
Block Diagram Representation of a System

- Differential equations can be used to represent relationship between input and output of a system.
- Problem: system parameters, and input ($r(t)$) and output ($c(t)$) appear throughout equation.
- Prefer to represent system as in Fig. 2.1(a) below where input and output are separate.

Figure 2.1

Want to be able to represent system as series of cascading subsystems, which can easily be combined together.

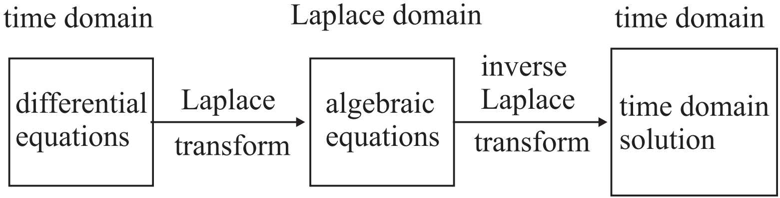
This can not be achieved with differential equations.



Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

Laplace Transform Review

- Helps us understand the dynamic behaviour of processes.
- Essential for: stability analysis, block diagrams, and controller design.
- Converts differential equations (time domain) into algebraic equations (s-domain).



Laplace Transform Definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s)$$

- $\mathcal{L}$ is the Laplace transform operator.
- $s = \sigma + j\omega$ is the Laplace transform variable and is a complex number.
- A sufficient condition for existence of Laplace transform is function is *piecewise continuous* in time over interval of integration.
- $F(s)$ contains no information about $f(t)$ for time $t < 0$.
- When doing the inverse Laplace transform, we can represent this by multiplying $f(t)$ by $u(t)$, the unit step function.

Common Laplace Transforms

Table 2.1

- Table shows the Laplace transform for several common signals.
- Rather than evaluating Laplace transform using its definition, common to use table and apply various theorems.

Item no.	f(t)	F(s)
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Important Properties of Laplace Transform

Linearity: $\mathcal{L}\{k_1 f_1(t) \pm k_2 f_2(t)\} = k_1 F_1(s) \pm k_2 F_2(s)$

Differentiation:

$$\mathcal{L}\{\frac{df(t)}{dt}\} = sF(s) - f(0-)$$

$$\mathcal{L}\{\frac{d^2 f(t)}{dt^2}\} = s^2 F(s) - sf(0-) - f'(0-)$$

Frequency Shifting: $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$

Laplace Transform Theorems

Table 2.2

Commonly used theorems.

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\,d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

[1] For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts and no more than one can be at the origin.
[2] For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (i.e., no impulses or their derivatives at $t = 0$).

The Transfer Function

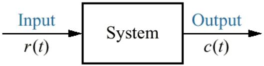
- We are now ready to represent system in the form of the diagram below.

- We will separate the system into the following distinct parts: system input, output, and transfer function.

- In general, an n^{th} order *linear, time-invariant (LTI)* differential equation is of the form:

$$
\begin{aligned}
a_n \frac{d^n c(t)}{dt^n} &+ a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) \\
&= b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)
\end{aligned}
$$

- $c(t)$ is output, $r(t)$ is input, and differential equation represents system's behavior.



The Transfer Function - II

- Taking Laplace transform of both sides, gives algebraic equation.

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{init terms for c(t)}$$
$$= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{init terms for r(t)}$$

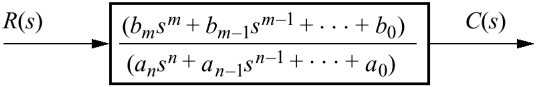
- If we assume initial conditions are zero, we can reduce this to:

$$\begin{aligned}(a_n s^n &+ a_{n-1} s^{n-1} + \cdots + a_0) C(s) \\ &= (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s)\end{aligned}$$

- $$\frac{C(s)}{R(s)} = G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

- $G(s)$ is called the transfer function.

- Can find ouput by $C(s) = R(s)G(s)$.



Transfer Function Example

- A given system can be represented by the following differential equation.

$$\tau_p \frac{dc(t)}{dt} + c(t) = k_p r(t)$$

- We will now derive a transfer function for the system.
- Taking the Laplace transform of both sides gives:

$$\mathcal{L}\{\tau_p \frac{dc(t)}{dt} + c(t)\} = \mathcal{L}\{k_p r(t)\}$$

- Using linearity, we get:

$$\tau_p \mathcal{L}\{\frac{dc(t)}{dt}\} + \mathcal{L}\{c(t)\} = k_p \mathcal{L}\{r(t)\} \qquad (1)$$

- Using differentiation theorem, we get:

$$\mathcal{L}\{\frac{dc(t)}{dt}\} = sC(s) - c(0-)$$

Transfer Function Example - II

- When applying Laplace transform to dynamic systems, we typically assume that all inputs are zero at $t < 0$, thus output responses will also be zero until $t = 0$.
- We can thus rewrite (1) as follows:

$$\tau_p \mathcal{L}\{\frac{dc(t)}{dt}\} + \mathcal{L}\{c(t)\} = k_p \mathcal{L}\{r(t)\}$$

$$\tau_p s C(s) - c(0-) + C(s) = k_p R(s)$$

$$\tau_p s C(s) + C(s) = k_p R(s)$$

$$\Rightarrow (\tau_p s + 1) C(s) = k_p R(s)$$

- Thus:

$$G(s) = \frac{C(s)}{R(s)} = \frac{k_p}{\tau_p s + 1} = \frac{\frac{k_p}{\tau_p}}{s + \frac{1}{\tau_p}}$$

Inverse Laplace Transform

- Once we have solved for the output as a function of the input in the s-domain, we typically want to convert this back into the time domain.

- We use the inverse Laplace transform:

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \lim_{\omega \to \infty} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s) e^{st} ds$$

- Normally, people look up $F(s)$ in tables, and make use of the theorems.

Partial Fraction Expansion

- The Laplace transform model of a system will typically be of form:

$$F(s) = \frac{N(s)}{D(s)}$$

- where $N(s)$ is an m^{th} order polynomial in s and $D(s)$ is an n^{th} order polynomial in s.

- For example, for $F(s)$ below $m = 2$ and $n = 5$:

$$F(s) = \frac{s^2 + 2s - 3}{s^5 + s^4 - s - 1}$$

- We know from algebra that all polynomials that have real coefficients can be factored into linear and irreducible quadratic factors.

- For example: $s^5 + s^4 - s - 1 = (s-1)(s+1)^2(s^2+1)$

Partial Fraction Expansion - II

- Using partial fraction decomposition and this factorization, we can write the transfer function as follows:

$$
\begin{aligned}
F(s) &= \frac{s^2+2s-3}{(s-1)(s+1)^2(s^2+1)} \\
&= \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{Ds+E}{s^2+1}
\end{aligned}
$$

- To complete the decomposition for this example, we would then determine values for A, B, C, D, and E that satisfy the equation.

- We can then apply the linearity and the frequency shifting properties to easily find the inverse Laplace transform.

Steps for Decomposing $\frac{N(s)}{D(s)}$ into Partial Fractions

1. **Divide if Improper:** If $\frac{N(s)}{D(s)}$ improper fraction (ie. degree of $N(s)$ is $\geq$ degree of $D(s)$) then divide $N(s)$ by $D(s)$ to get

$$\frac{N(s)}{D(s)} = \text{(a polynomial)} + \frac{N_1(s)}{D(s)}$$

where $\frac{N_1(s)}{D(s)}$ is not an improper fraction. We would then apply the following steps to $\frac{N_1(s)}{D(s)}$.

2. **Factor Denominator:** Factor denominator into factors of form:

$$(ps+q)^m \qquad \text{and} \qquad (as^2+bs+c)^n$$

where $as^2 + bs + c$ can not be further reduced.

Steps for Decomposing $\frac{N(s)}{D(s)}$ - II

3. **Linear Factors:** For factor $(ps+q)^m$, include the sum below:

$$\frac{A_1}{(ps+q)} + \frac{A_2}{(ps+q)^2} + \cdots + \frac{A_m}{(ps+q)^m}$$

4. **Quadratic Factors:** For factor $(as^2+bs+c)^n$, include the sum below:

$$\frac{B_1 s + C_1}{(as^2+bs+c)} + \frac{B_2 s + C_2}{(as^2+bs+c)^2} + \cdots + \frac{B_n s + C_n}{(as^2+bs+c)^n}$$

5. **Determine Unkowns:** Equate original fraction to the sums found in steps 3 and 4, and solve for unknowns.

Quadratic Factors e.g.

Find: $\mathcal{L}^{-1}\{\frac{3}{s(s^2+2s+5)}\}$

For a quadratic of the form $as^2 + bs + c$, we can use the quadratic equation to determine its factors.

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For us, this gives complex factors, thus the term is irreducible.

$$s = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = -1 \pm 2j$$

$$\frac{3}{s(s^2 + 2s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 5}$$

Quadratic Factors e.g. - II

Basic Equation:

$$3 = A(s^2 + 2s + 5) + s(Bs + C) = (A + B)s^2 + (2A + C)s + 5A$$

For $s = 0$, we get: $3 = 5A$ thus $A = \frac{3}{5}$

Substituting A in gives: $3 = (\frac{3}{5} + B)s^2 + (\frac{6}{5} + C)s + 3$

Equating coefficients of powers of s gives:

$$\left(\frac{3}{5} + B\right) = 0 \qquad \text{and} \qquad \left(\frac{6}{5} + C\right) = 0$$

We thus have $B = -\frac{3}{5}$ and $C = -\frac{6}{5}$.

We thus have:

$$\frac{3}{s(s^2 + 2s + 5)} = \frac{\frac{3}{5}}{s} + \frac{-\frac{3}{5}s + -\frac{6}{5}}{s^2 + 2s + 5} = \frac{\frac{3}{5}}{s} + \left(-\frac{3}{5}\right)\frac{s + 2}{s^2 + 2s + 5}$$

Quadratic Factors e.g. - III

The quadratic term is the sum of the Laplace transforms of frequency shifted sine and cosine.

This gives:
$\mathcal{L}\{K_1 e^{-at} cos\omega t\} = \frac{K_1(s+a)}{(s+a)^2+\omega^2}$ $\mathcal{L}\{K_2 e^{-at} sin\omega t\} = \frac{K_2\omega}{(s+a)^2+\omega^2}$.

Adding the transforms together, we get:

$$\mathcal{L}\{K_1 e^{-at} cos\omega t + K_2 e^{-at} sin\omega t\} = \frac{K_1(s+a) + K_2\omega}{(s+a)^2 + \omega^2}$$

We next note that $(s+a)^2 + \omega^2 = s^2 + 2as + (a^2 + \omega^2)$

Equating to our denominator $(s^2 + 2s + 5)$, gives: $2a = 2$ and $(a^2 + \omega^2) = 5$

We thus have $a = 1$ and $\omega = 2$.

Quadratic Factors e.g. - IV

We now equate our numerators and get:

$$s + 2 = K_1(s + 1) + 2K_2 = K_1 s + (K_1 + 2K_2)$$

This gives us $K_1 = 1$ and $K_2 = \frac{1}{2}$

We thus have:

$$\frac{s + 2}{s^2 + 2s + 5} = \frac{(s + 1) + (\frac{1}{2})2}{(s + 1)^2 + 2^2}$$

Putting it all together:

$$\mathcal{L}^{-1}\{\frac{\frac{3}{5}}{s} + (-\frac{3}{5})\frac{s + 2}{s^2 + 2s + 5}\} = [\frac{3}{5} - \frac{3}{5}e^{-t}(cos2t + \frac{1}{2}sin2t)]u(t)$$

We note that the factors of $s^2 + 2s + 5$ were $s = -1 \pm 2j$
$= -a \pm \omega j$.

Stability Analysis Using Roots of $D(s)$

- The roots of $D(s)$ correspond to the exponential terms in the time domain response for $G(s)$, the transfer function.

$$
\begin{aligned}
G(s) &= \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \\
&= \frac{A_1}{(s+p_1)} + \frac{A_2}{(s+p_2)} + \cdots + \frac{A_n}{(s+p_n)}
\end{aligned}
$$

We refer to the roots of $D(s)$ as poles.

Solving for $g(t)$ gives:

$$
\begin{aligned}
g(t) &= \mathcal{L}^{-1}\left\{\frac{A_1}{(s+p_1)}\right\} + \mathcal{L}^{-1}\left\{\frac{A_2}{(s+p_2)}\right\} + \cdots + \mathcal{L}^{-1}\left\{\frac{A_n}{(s+p_n)}\right\} \\
&= A_1 e^{-p_1 t} + A_2 e^{-p_2 t} + \cdots + A_n e^{-p_n t}
\end{aligned}
$$

Stability Analysis: Real Roots

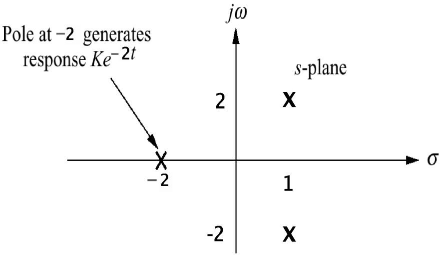
Considering: $g(t) = A_1 e^{-p_1 t} + A_2 e^{-p_2 t} + \cdots + A_n e^{-p_n t}$

If we have pole(s) at $s = -p_i = -\sigma_i \pm j\omega_i$:

- If $\omega_i = 0$, then pole is strictly real and corresponds to $A_i e^{-\sigma_i t}$

  If $\sigma_i > 0$, then pole in left side of imaginary plane, and response decreases to zero over time and system is stable.

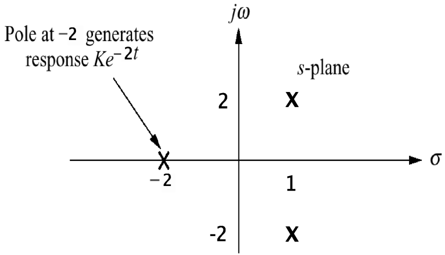
i.e. $\sigma_i = 2$, then we get $s = -2$, $p_i = 2$ and response $A_i e^{-\sigma_i t} = A_i e^{-2t}$.



Stability Analysis: Real and Imaginary Roots

- If $\omega_i = 0$ and $\sigma_i < 0$, then pole in right side of imaginary plane, and response increases over time and system is unstable. i.e. $\sigma_i = -1$, then we get $s = 1$, $p_i = -1$ and response $A_i e^{-\sigma_i t} = A_i e^t$.
- If $\sigma_i = 0$ and $\omega_i \neq 0$, then we have two pure imaginary roots. This corresponds to a sinusoidal response with no damping, technically considered stable.

i.e. $\omega_i = 2$, then we get $s = \pm 2j$, $p_i = \pm 2j$ and response $A_i sin\omega_i t = A_i sin2t$.



Stability Analysis: Complex Roots

If we have $\sigma_i \neq 0$ and $\omega_i \neq 0$, we have complex roots.

- For poles at $s = -\sigma_i \pm j\omega_i$ we get the partial fraction expansion below:

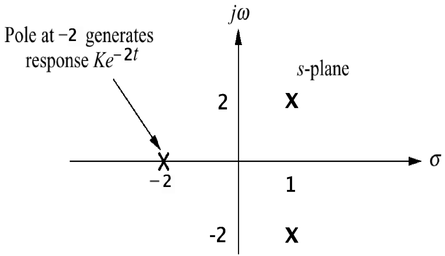
$$\frac{\alpha + j\beta}{s + \sigma_i + j\omega_i} + \frac{\alpha - j\beta}{s + \sigma_i - j\omega_i}$$

- This results in the time domain response of the form:

$$e^{-\sigma_i t}[2\alpha\, cos\omega_i t + 2\beta\, sin\omega_i t]$$

If $\sigma_i > 0$, then response is stable.

i.e. $\sigma_i = -2$ and $\omega_i = 3$, then we get $s = 2 \pm 3j$, $p_i = -2 \pm 3j$ and response $e^{-\sigma_i t}[2\alpha\, cos\omega_i t + 2\beta\, sin\omega_i t]$ $= e^{2t}[2\alpha\, cos3t + 2\beta\, sin3t]$.



Time Functions Associated with s-plane

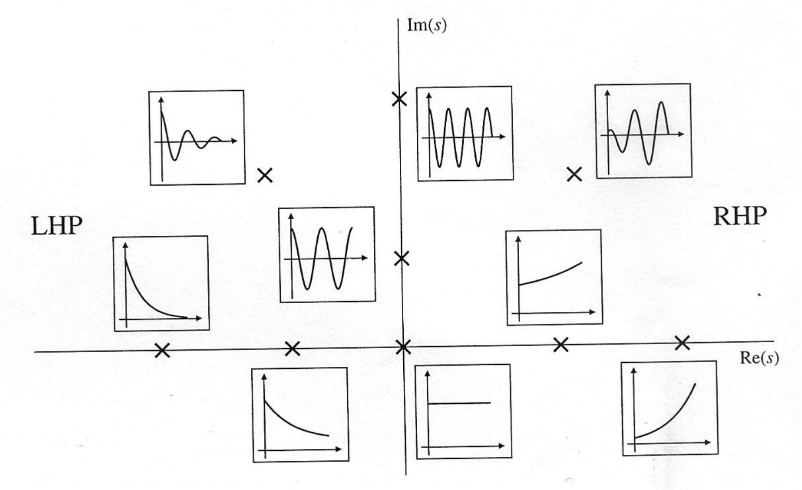
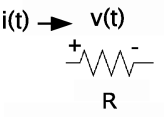


Figure 2.5 from Dorf and Bishop, *Modern Control Systems (10th Edition)*, Prentice-Hall, 2004.

Electric Network Transfer Function

- We will generalize notion of resistance to impedance.
- This will allow us to treat capacitors and inductors in similar manner as resistors in analyzing circuits.
- Recall that Ohms law says: $v(t) = Ri(t)$
- Taking Laplace Transform of both sides gives: $V(s) = RI(s)$
- We define the impedance to be $Z(s) = \frac{V(s)}{I(s)} = R$
- We also define the admittance to be: $Y(s) = \frac{I(s)}{V(s)} = \frac{1}{R} = G$

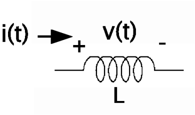


Impedance of Inductor

- The voltage-current relation for an inductor is: $v(t) = L\frac{di(t)}{dt}$.
- To find transfer function, take Laplace transform of each side and assume zero initial conditions:

$$V(s) = LsI(s)$$

- The impedance is thus $Z(s) = \frac{V(s)}{I(s)} = Ls$



Impedance of Capacitor

- The voltage-current relation for a capacitor is:

$$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

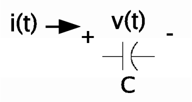
- Taking derivative of both sides gives:

$$\frac{dv(t)}{dt} = \frac{1}{C}i(t)$$

- Taking Laplace transform (assuming zero initial conditions) gives:

$$sV(s) = \frac{1}{C}I(s)$$

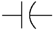
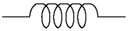
- The impedance is thus $Z(s) = \frac{V(s)}{I(s)} = \frac{1}{Cs}$



Summary of Circuit Elements.

- Table 2.3 below shows for each element type the voltage-current, current-voltage, and voltage-charge relationship, as well as the elements impedance and admittance.

- All elements here are passive as they do not contain an internal energy source.

 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

Equivalent Resistance and Impedance.

- It can be shown that resistance in serial can be replaced by an equivalent resistor with resistance R_s calculated as follows:

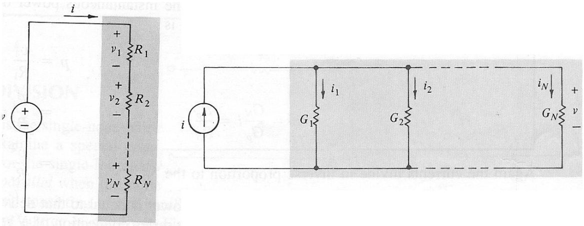
$$R_s = R_1 + R_2 + \cdots + R_N$$

- This can be generalized to impedance as follows:

$$Z_s = Z_1 + Z_2 + \cdots + Z_N$$

- The equivalent resistance (R_p) and impedance (Z_p) when elements are connected in parallel is:

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N} \qquad \frac{1}{Z_p} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N}$$



Figures 2.16 and 2.18 from *Electric Circuit Analysis*, D.E. Jounson et al., 1989.

Equivalent Resistance Example

Figure (a) to (b): $R_{s_1} = 1 + 5 = 6\Omega$

Figure (b) to (c): $\frac{1}{R_p} = \frac{1}{12} + \frac{1}{6} \Rightarrow R_p = 4\Omega$

Figure (c) to (d): $R_{s_2} = 7 + 4 = 11\Omega$

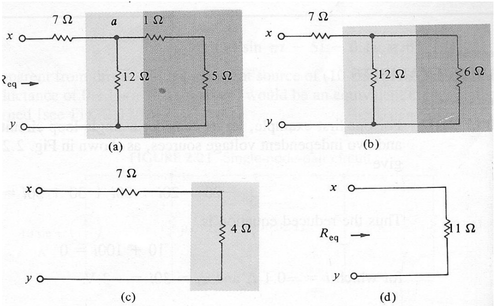
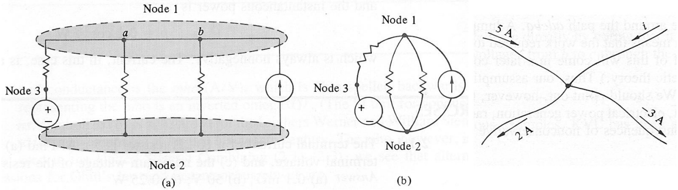


Figure 2.19 from *Electric Circuit Analysis*, D.E. Jounson et al., 1989.

Kirchhoff's Current Law

- A point of connection between two or more circuit elements is referred to as a node.
- Circuit show in Fig. (a) contains three nodes and is electrically equivalent to that of Figure (b).
- Kirchoff's current law (KCL) says that *the algebraic sum of the currents entering any node is zero.*
- Currents flowing into the node are considered positive, and those leaving negative.
- For node on right, this would give: $5 + i - (-3) - 2 = 0$



Figures 2.8 and 2.9 from *Electric Circuit Analysis*, D.E. Jounson et al., 1989.

Kirchhoff's Voltage Law

- Kirchoff's voltage law (KVL) says that *the algebraic sum of voltages around any closed path is zero.*

- In the direction we traverse the path, voltages that go from $-$ to $+$ (lower to higher potential) are considered to be positive, and those going from $+$ to $-$ to be negative.

- Traversing the circuit below in clockwise direction gives:

$$5 - v - 10 + 2 = 0$$

- Solving for v gives $v = -3V$.

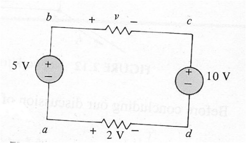
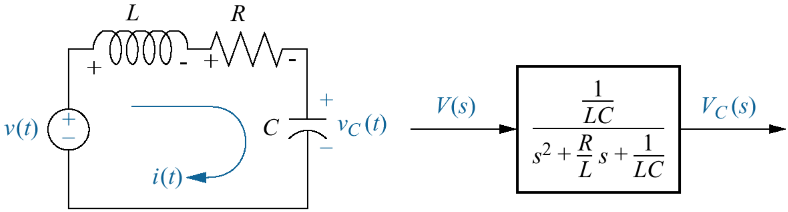


Figure 2.11 from *Electric Circuit Analysis*, D.E. Jounson et al., 1989.

Kirchhoff's Voltage Law eg.

- Find transfer function relation $\frac{V_C(s)}{V(s)}$.



Figures 2.4 and 2.5.

Mesh Analysis

- Replace passive elements with their impedances, and all sources and time variables with their Laplace transform.
- Assume a transform current and direction in each mesh.
- Assume for each element a voltage polarity
- Assume a current $I_3(s)$ and direction through the shared segment (ie. inductor)
- Apply Kirchoff's voltage law to each mesh going in direction of the mesh current.
- Use Kirchoff's current law to relate the currents.

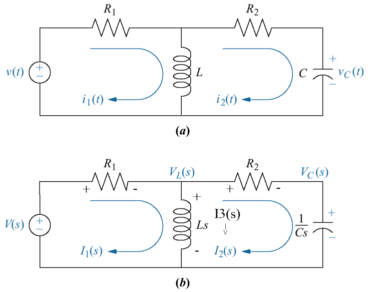
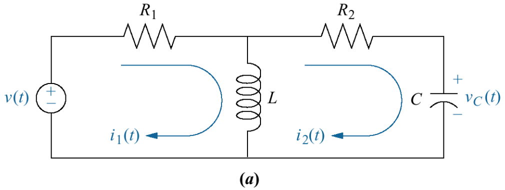


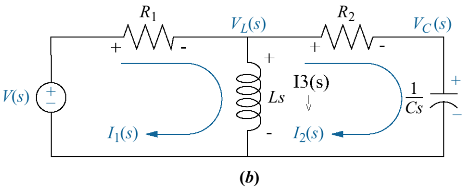
Figure 2.6.

Mesh Analysis eg.

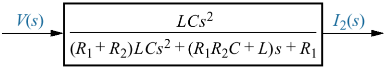
- Find transfer function relation $\frac{I_2(s)}{V(s)}$.



(a)



(b)



$$\frac{LCs^2}{(R_1+R_2)LCs^2+(R_1R_2C+L)s+R_1}$$

(c)

Figures 2.6.

Cramer's Rule

Cramer's Rule: *If* $Ax = B$ *is a system of* n *linear equations in* n *unknowns such that det*$(A) \neq 0$*, then the system has a unique solution.* The solution is:

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)}, \cdots, x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries of the j^{th} column of A by the entries in the matrix B.

Definition taken from *Elementary Linear Algebra with Applications* by H. Anton et al. , 1987.

Matrix Determinant

The formula of a determinant for a 2×2 and a 3×3 matrix:

$$\text{(i)} \quad \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\text{(ii)} \quad \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Definition taken from *Elementary Linear Algebra with Applications* by H. Anton et al. , 1987.

Mnemonic device for remembering determinant formula.

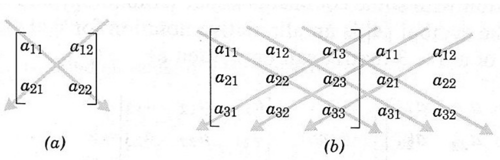


Figure 2.2 from *Elementary Linear Algebra with Applications* by H. Anton et al. , 1987.

Nodal Analysis

- Transform circuit into Laplace Domain.
- Assume for each element a voltage polarity.
- Determine nodes for circuit. Choose one (treat like ground) - others relative to it.
- Assume transform currents and directions entering and leaving each node.

- Use Kirchoff's current law to create equations for each node with an unknown voltage.
- Use relation $I(s) = \frac{V(s)}{Z(s)}$ to get equation for currents in terms of voltages.

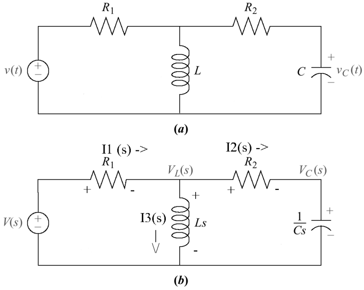
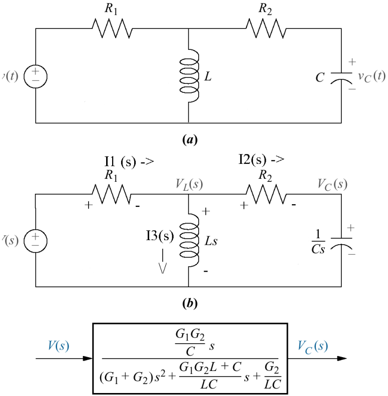


Figure 2.6.

Nodal Analysis eg.

- Find transfer function relation $\frac{V_C(s)}{V(s)}$.



Figures 2.6 and 2.7.

Operational Amplifiers

- So far we have only used passive elements for transfer functions.
- We can use operational amplifiers (op amps) to construct active circuits that can be used as transfer functions.
- Op amps have the following characteristics:

1. Output: $v_o(t) = A(v_2(t) - v_1(t))$
2. High input impedance: $Z_i = \infty$ (ideal)
3. Low output impedance: $Z_o = 0$ (ideal)
4. High constant gain: $A = \infty$ (ideal)

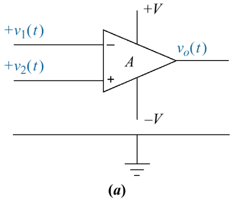


Figure 2.10.

In linear operating region "ideal Op Amp" assumptions:

(i) Input voltages are equal: $v_1(t) = v_2(t)$

(ii) No current flows into inputs: $i_+ = i_- = 0$

Inverting Op Amps

- If we tie $v_2(t)$ to ground, we get an inverting op amp, with output $v_o(t) = -Av_1(t)$.
- Usually use with feedback in form of figure on the right.
- As input impedance very large, $I_a(s) \approx 0$.
- Using nodal analysis at v_1, we find $I_1(s) = -I_2(s)$.
- If gain A very large, feedback forces $v_1(t) \approx 0$.

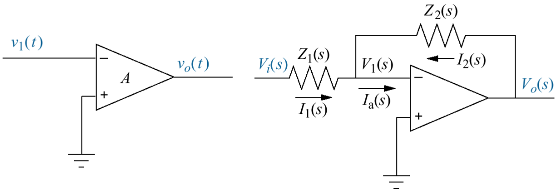


Figure2.10.

Inverting Op Amps - II

- With $v_1(t) \approx 0$, we thus have $I_1(s) = \frac{V_i(s)}{Z_1(s)}$ and $I_2(s) = \frac{V_o(s)}{Z_2(s)}$.
- Substituting into eqn $I_1(s) = -I_2(s)$ gives:

$$\frac{V_i(s)}{Z_1(s)} = -\frac{V_o(s)}{Z_2(s)}$$

- This gives the transfer function:

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

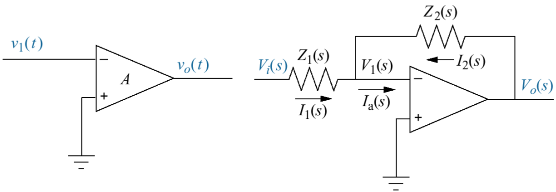


Figure2.10.

Inverting Op Amp eg.

- Find transfer function $\frac{V_o(s)}{V_i(s)}$.

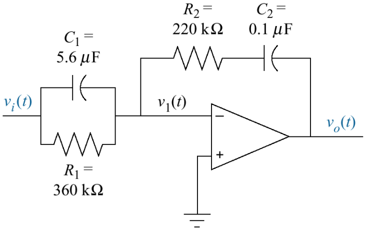


Figure 2.11.

Formula for inverting op amp is:

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} \tag{2}$$

We first need to determine $Z_1(s)$ and $Z_2(s)$ to use formula.

Using parallel inductance rule, we get:

$$\frac{1}{Z_1(s)} = \frac{1}{1/C_1 s} + \frac{1}{R_1} = C_1 s + \frac{1}{R_1}$$

Inverting Op Amp eg. - II

Which gives eqn below:

$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6 \times 10^{-6} s + \frac{1}{360 \times 10^3}} = \frac{360 \times 10^3}{2.016s + 1}$$

Using serial inductance rule, we get:

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s}$$

Substituting into equation 2, we get:

$$\frac{V_o(s)}{V_i(s)} = -\frac{220 \times 10^3 + \frac{10^7}{s}}{\frac{360 \times 10^3}{2.016s+1}}$$

Multiplying top and bottom by $(2.016s + 1)$ and simplifying gives:

$$-\frac{443,520s + 20,380,000 + \frac{10^7}{s}}{360 \times 10^3}$$

Inverting Op Amp eg. - III

$$-\frac{443,520s + 20,380,000 + \frac{10^7}{s}}{360 \times 10^3}$$

Multiply top and bottom by s and factoring out $\frac{443{,}520}{360\times 10^3} = 1.232$ gives:

$$-1.232[\frac{s^2 + 45.95s + 22.55}{s}]$$

We note that our equation is in the form of a PID controller, as shown below.

Transfer function of a PID controller:

$$G_c(s) = \frac{K_3(s^2 + \frac{K_1}{K_3}s + \frac{K_2}{K_3})}{s}$$

Noninverting Op Amp

- We know that $V_o(s) = A(V_i(s) - V_1(s))$
- Using voltage division, we can derive:

$$V_1(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_o(s)$$

- Substituting into top equation for $V_1(s)$, gives:

$$\frac{V_o(s)}{V_i(s)} = \frac{A}{1 + \frac{AZ_1(s)}{Z_1(s)+Z_2(s)}}$$

- As A is large, we can disregard the 1 term in denominator, which gives:

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$$

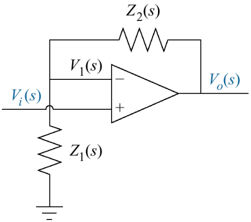


Figure 2.12.

Mechanical Systems

- We now examine how to relate the Laplace transform of the input to that of the output using a transfer function.

- We will consider two types of mechanical systems:
  1. translational mechanical systems.
  2. rotational mechanical systems.

- As then end result is a transfer function that is mathematically indistinguishable from that of an electrical network, we can thus connect the two by cascading their transfer functions.

Mass Component

- Newton's Second Law of Motion: $\Sigma f = Ma$

- We are interested in the following properties of the object:
  1. $\Sigma f = f(t)$**:** the summation of the forces acting on the object.
  2. M**:** the object's mass.
  3. $x(t)$**:** the position of the object.
  4. $v(t)$**:** the velocity of the object which is $\frac{dx}{dt}$.
  5. $a(t)$**:** the acceleration of the object which is $\frac{d^2x}{dt^2}$.

- We wish to find the transfer function $Z_m(s) = \frac{F(s)}{X(s)}$.

- We have: $f(t) = Ma(t) = M\frac{d^2x}{dt^2}$.
- Taking the Laplace transform of both sides gives: $F(s) = Ms^2X(s)$.
- Thus $Z_m(s) = \frac{F(s)}{X(s)} = Ms^2$.
- Similarly, $Z'_m(s) = \frac{F(s)}{V(s)} = Ms$

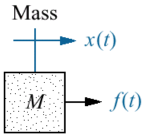


Table 2.4

Viscous Damper

- Middle part is typically moving through some sort of fluid.
- Frictional force generated is proportional to object's velocity:

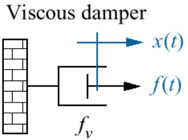
$$f(t) = f_v\, v(t)$$

where f_v is called *coefficient of viscous friction*.

- Substituting $v(t) = \frac{dx}{dt}$ gives

$$f(t) = f_v \frac{dx}{dt}$$

- Taking the Laplace transform of both sides gives: $F(s) = f_v\, sX(s)$.



- Thus $Z_m(s) = \frac{F(s)}{X(s)} = f_v s$.
- Similarly, $Z'_m(s) = \frac{F(s)}{V(s)} = f_v$

Table 2.4.

Spring Component

- The *spring constant* is denoted K.

- Spring's force is proportional to distance x which gives us:

$$f(t) = Kx(t)$$

- Taking the Laplace transform of both sides gives:

$$F(s) = KX(s)$$

- We thus have: $Z_m(s) = \frac{F(s)}{X(s)} = K$.

- With zero initial conditions, we have: $x(t) = \int_0^t v(\tau)d\tau$
- Thus $F(s) = \frac{K}{s}V(s)$
- This gives: $Z'_m(s) = \frac{F(s)}{V(s)} = \frac{K}{s}$

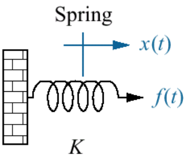
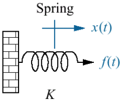
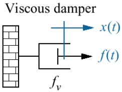
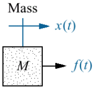


Table 2.4.

Summary of Translational Elements

- Table gives Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass.

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

Table 2.4.

Translational eg. - one equation of motion

- Find transfer function $\frac{X(s)}{F(s)}$
- $f(t)$ is the force we are applying in the direction of $x(t)$.
- To solve for transfer function, do the following:
  1. Draw free body drawing showing all forces acting on object, and their directions.
  2. Replace mechanical components with mechanical impedances.
  3. Newton's law says that $Ma(t)$ equals the sum of the forces acting on the object. Use this to create your force equation.

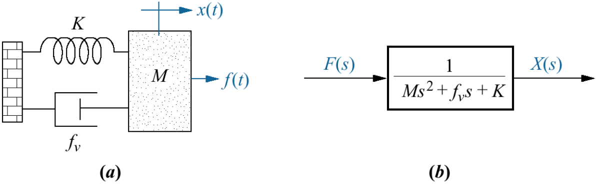
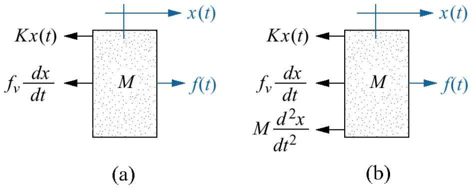


Figure 2.15.

Translational eg. - one equation of motion - II

- We assume the mass is travelling to the right.
- This means only the applied force ($f(t)$) points to the right.
- The spring and viscous damping forces impede the motion, thus act to oppose it.
- Newton's second law: $M\frac{d^2x}{dt^2} = f(t) - f_v\frac{dx}{dt} - Kx(t)$
- Rearranging this gives: $f(t) = M\frac{d^2x}{dt^2} + f_v\frac{dx}{dt} + Kx(t)$

- (a) Proper application of Newton's law.
- (b) Text book's "method."



Free body diagram..

Translational eg. - one equation of motion - III

- Can think of the $Ma(t)$ term as force beyond spring and viscous damping forces, needed to attain acceleration $a(t)$.

- Using textbook's method, forces acting in assumed direction of motion are positive, those acting in opposite are negative, and they should sum to zero.

- (a) Free-body diagram of mass, spring, and damper system.

- (b) Transformed free-body diagram.

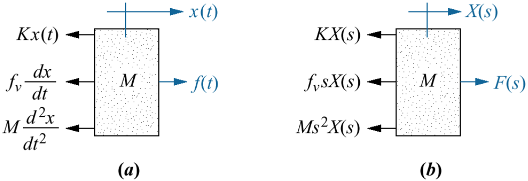


Figure 2.16.

Translational eg. - two degrees of freedom

- Number of equations of motion needed = number of linearly independent motions.

- A linearly independence means that one object can still move when all other objects are held still.

- We also refer to number of linearly independent motions as the number of degrees of freedom of the system.

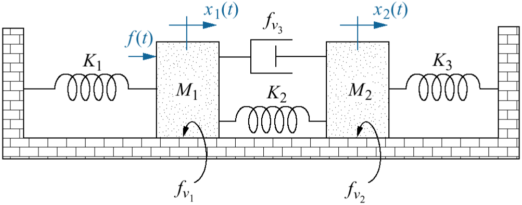


Figure 2.17 (a).

Translational eg. - two degrees of freedom - II

- Find transfer function $\frac{X_2(s)}{F(s)}$
- To solve a problem like this, we will use the principal of superposition. To find the forces acting on a given object (say M_1), we do the following:
  1. Draw the free body diagram for object (ie. M_1) by holding all other objects (ie. M_2) still, and finding the forces acting on the object (ie. M_1) due only to its motion.
  2. We then hold the object (ie. M_1) still, and activate each other object (ie. M_2) one at a time, and determine the forces acting on original object (ie. M_1) due to the other object's motion.
  3. The total force acting on the original object is the superposition of the forces found in steps 1 and 2.

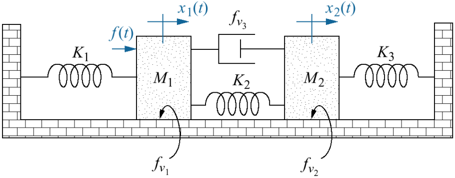


Figure 2.17 (a).

Translational eg. - two degrees of freedom - III

- Use Newton's law to find equation of motion for M_1.

- (a) Forces on M_1 due to only motion of M_1.
- (b) Forces on M_1 due to only motion of M_2.
- (c) All forces on M_1.

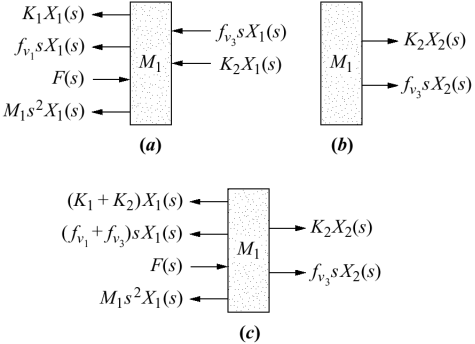


Figure 2.18.

Translational eg. - two degrees of freedom - IV

- Use Newton's law to find equation of motion for M_2.

- (a) Forces on M_2 due to only motion of M_2.
- (b) Forces on M_1 due to only motion of M_1.
- (c) All forces on M_2.

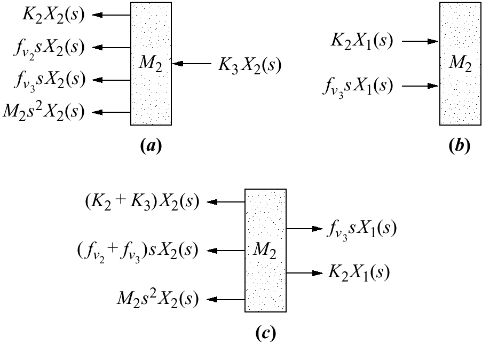


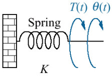
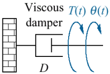
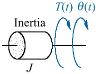
Figure 2.19.

Rotational Mechanical Systems

- With rotational mechanical systems, we are dealing with objects that rotate about a fixed axis.
- The main differences from a translational system are as follows:
  - $x(t)$ displacement becomes $\theta(t)$ angular displacement.
  - $v(t) = \frac{dx(t)}{dt}$ velocity becomes $\omega(t) = \frac{d\theta(t)}{dt}$ angular velocity.
  - $f(t)$ force becomes $T(t)$ torque.
  - M mass becomes J moment of inertia.
  - f_v coefficient of viscous damping becomes D coefficient of viscous angular damping.
  - K spring constant becomes K angular spring constant.

Summary of Rotational Components

- Same symbols for components, but undergoing rotation.
- We are interested in the relation: $T(s) = Z_m(s)\theta(s)$

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)\,d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ = N-m (newton-meters), $\theta(t)$ = rad (radians), $\omega(t)$ = rad/s (radians/second), K = N-m/rad (newton-meters/radian), D = N-m-s/rad (newton-meters-seconds/radian), J = kg-m^2 (kilogram-meters^2 = newton-meters-seconds^2/radian).

Table 2.5.

Would have for $Z'_m(s) = \frac{T(s)}{\omega(s)}$ the values $\frac{K}{s}$, D, and Js.

Degrees of Freedom

- Similar idea as for translation systems.
- The degrees of freedom is the number of objects (*points of motions*) that can still rotate when all other objects are held still.
- The degrees of freedom equals the number of equations of motion we need to describe a system.

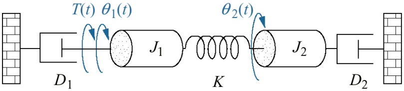


Figure 2.22 (b).

Writing Equations of Motion - Rotational

- Create free body diagram, but show torques and angular displacement instead.

- We will use superposition. To find the torques action on J_1:
  1. Draw the free body diagram for J_1 by holding J_2 still, and find the torques acting on J_1 due only to its own motion.
  2. We then hold J_1 still, and rotate J_2, and determine the torques acting on J_1 due to the motion of J_2.
  3. The total torque acting on J_1 is the superposition of the torques found in steps 1 and 2.
- Repeat above, but for J_2.

- The sum of torques acting in direction of assumed rotation equals the sum of the torques opposing.

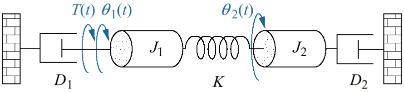


Figure 2.22 (b).

Rotational eg.

- Find the transfer function $\frac{\theta_2(S)}{T(S)}$.
- We have a flexible rod being supported at both ends by bearings, and is undergoing torsion.
- We assume system can be approximated by a spring at one point in the rod, with an inertia of J_1 on the left, and J_2 on the right.

- (a) Physical system.
- (b) Schematic.

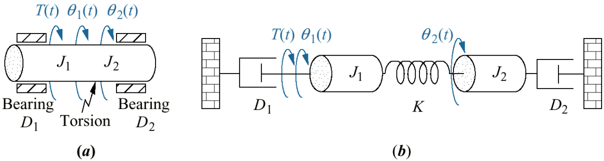


Figure 2.22.

Free body Diagrams for J_1

- (a) Torques on J_1 due only to motion of J_1.
- (b) Torques on J_1 due only to motion of J_2.
- (c) All Torques on J_1.

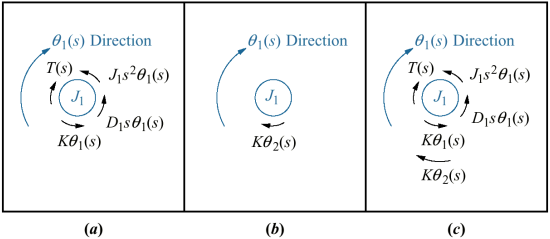


Figure 2.23.

We now sum torques opposing rotation and equate that to the torques in direction of rotation giving:

$$J_1 s^2\theta_1(s) + D_1 s\theta_1(s) + K\theta_1(s) = T(s) + K\theta_2(s) \qquad (3)$$

Free body Diagrams for J_2

- (a) Torques on J_2 due only to motion of J_2.
- (b) Torques on J_2 due only to motion of J_1.
- (c) All torques on J_2.

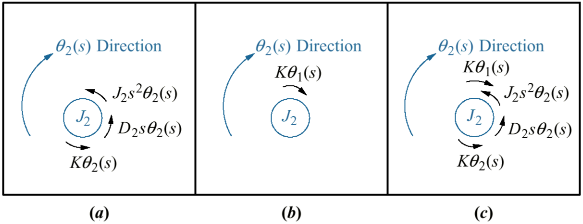


Figure 2.24.

We now sum torques opposing rotation and equate that to the torques in direction of rotation giving:

$$J_2 s^2 \theta_2(s) + D_2 s \theta_2(s) + K\theta_2(s) = K\theta_1(s) \qquad (4)$$

Rotational eg. - II

For equations 3 and 4, we move unknowns to the left side, and collect terms for θ_1 and θ_2 giving:

$$(J_1 s^2 + D_1 s + K)\theta_1(s) - K\theta_2(s) = T(s) \tag{5}$$

$$-K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) = 0 \tag{6}$$

Putting this in the form of $\underline{A}\,\underline{x} = \underline{B}$ gives:

$$\underline{x} = \begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix}; \quad \underline{A} = \begin{bmatrix} J_1 s^2 + D_1 s + K & -K \\ -K & J_2 s^2 + D_2 s + K \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} T(s) \\ 0 \end{bmatrix};$$

Rotational eg. - III

Cramer's rule given below where $\underline{A_2}$ is $\underline{A}$ with second column replace with $\underline{B}$:

$$\theta_2(s) = \frac{|\underline{A_2}|}{|\underline{A}|} = \frac{\left|\begin{bmatrix} J_1 s^2 + D_1 s + K & T(s) \\ -K & 0 \end{bmatrix}\right|}{|\underline{A}|} = \frac{KT(s)}{|\underline{A}|}$$

We thus have:

$$\frac{\theta_2(s)}{T(s)} = \frac{K}{|\underline{A}|}$$

Rotational Mechanical System with Gears

- Usually when rotational systems are driven by motors, we find associated gear trains driving the load.

- In diagram below, we see that if we apply torque $T_1(t)$ to our input gear which has N_1 teeth and radius r_1, we will get a corresponding torque $T_2(t)$ out of gear 2, which has N_2 teeth, and radius r_2.

- We want to determine the ratios $\frac{\theta_2}{\theta_1}$ and $\frac{T_2}{T_1}$.

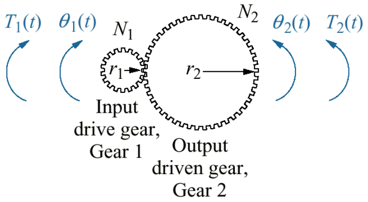


Figure 2.27.

Transfer Functions for Gears

- When gears turn, distance travelled around each gear's circumference is equal.
- We thus have: $r_1\theta_1 = r_2\theta_2 \Rightarrow \frac{\theta_2}{\theta_1} = \frac{r_1}{r_2}$
- It can be shown that $\frac{r_1}{r_2} = \frac{N_1}{N_2}$
- We thus have: $\frac{\theta_2}{\theta_1} = \frac{N_1}{N_2}$

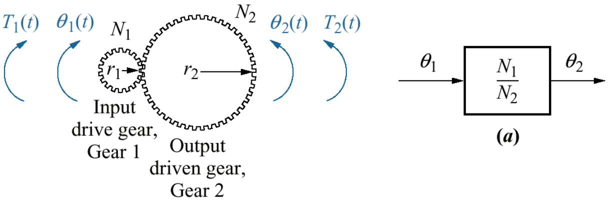


Figure 2.27 and Figure 2.28.

Transfer Functions for Gears - lossless

- If we assume the gears have negligible inertia and damping, then the energy put into gear 1 equals energy out of gear 2.
- We thus have: $T_1\theta_1 = T_2\theta_2 \Rightarrow \frac{T_2}{T_1} = \frac{\theta_1}{\theta_2}$
- Substituting in transfer function: $\frac{\theta_2}{\theta_1} = \frac{N_1}{N_2}$
- We get: $\frac{T_2}{T_1} = \frac{N_2}{N_1}$

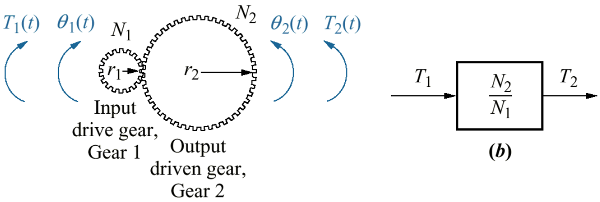


Figure 2.27 and Figure 2.28.

Using Transfer Function for Lossless Gears

- Our first step is to reflect T_1 to the other side of output gear giving: $T_1(s)\frac{N_2}{N_1}$.

- This gives equation of motion:

$$(Js^2 + Ds + K)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

- This would allows us to find: $\frac{\theta_2(s)}{T_1(s)}$

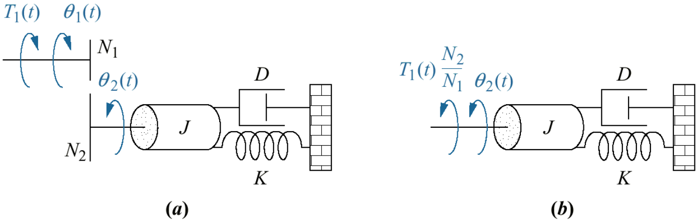


Figure 2.29.

Lossless Gears: reflecting impedance

- Substituting in for $\theta_2(s)$ gives

$$(Js^2 + Ds + K)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1}$$

- Simplifying: $[J(\frac{N_1}{N_2})^2 s^2 + D(\frac{N_1}{N_2})^2 s + K(\frac{N_1}{N_2})^2]\theta_1(s) = T_1(s)$

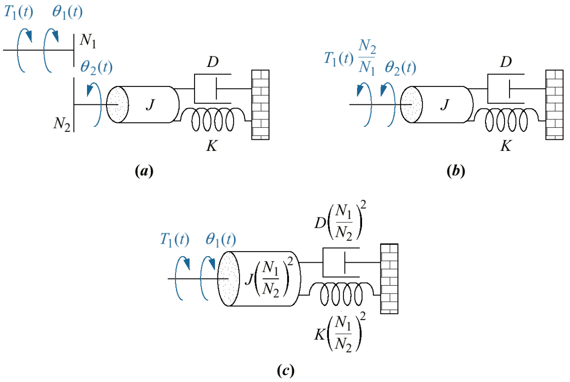


Figure 2.29.

Generalizing Reflecting Impedance

- **Rule:** *"Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio:"*

$$\left(\frac{\text{Number of teeth of gear on destination shaft}}{\text{Number of teeth of gear on source shaft}}\right)^2$$

- The impedance to be reflected is attached to the source shaft and is being reflected to the destination shaft.

- In previous example gear 2 was the source shaft, and gear 1 was the destination shaft.

Lossless Gears eg.

- Find transfer function: $\frac{\theta_2(s)}{T_1(s)}$

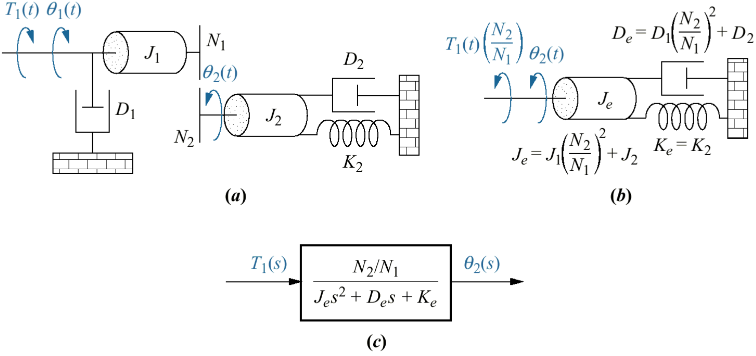


Figure 2.30.

Gear Trains

- What if you wanted a high ratio, say $\frac{100}{1}$?
- To avoid a gear with $100\times$ as big radius, a gear train is used.
- The total equivalent gear ratio for the gear train, is the product of the gear ratios for each step.

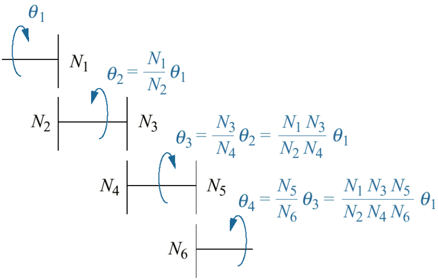


Figure 2.31.

For torques, we would have: $\frac{T_4}{T_1} = \frac{N_2 N_4 N_6}{N_1 N_3 N_5}$

Gears With Loss eg.

- Find transfer function $\frac{\theta_1(s)}{T_1(s)}$.
- Each gear below has its own inertia, and some shafts have nonnegligible damping.

- Need to reflect all moments of inertial and viscous dampers to the left of gear 1.

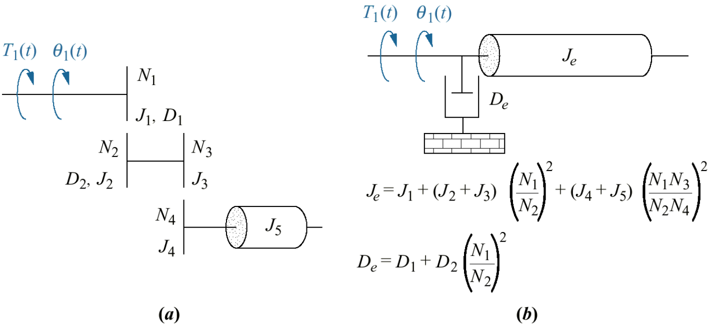


Figure 2.32.

Gears With Loss eg. - II

- Equation of motion:

$$(J_e s^2 + D_e s)\theta_1(s) = T_1(s)$$

- The transfer function is thus:

$$G(s) = \frac{\theta_1(s)}{T_1(s)} = \frac{1}{J_e s^2 + D_e s}$$

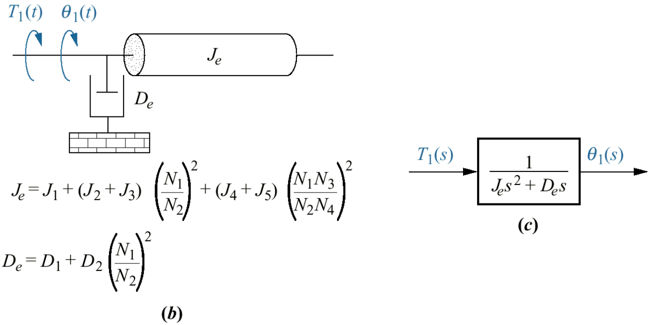


Figure 2.32.

Electromechanical System Transfer System

- Electromechanical systems are systems with a mixture of electrical and mechanical variables.

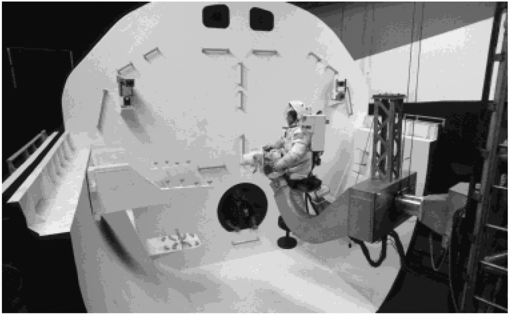


Figure 2.34: NASA flight simulator robot arm with electromechanical control system components.

DC Motors

- A motor takes a voltage as input and produces a physical displacement as output.
- We will derive the transfer function for the *armature-controlled dc servomotor.*

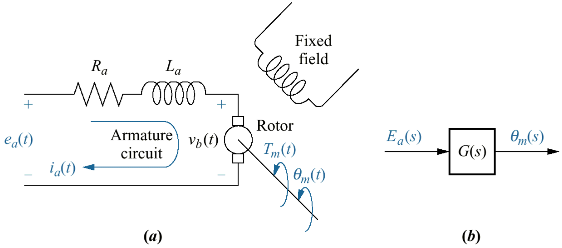
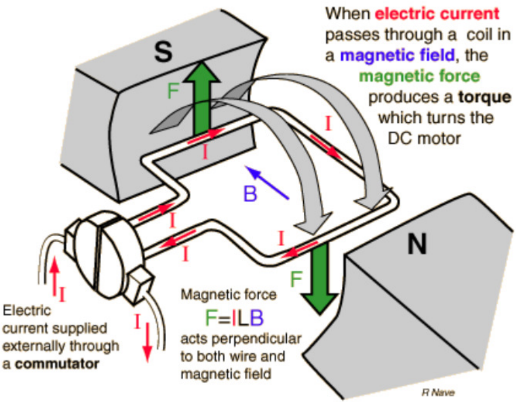


Figure 2.35.

DC Motor Principles



From: http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/motdc.html

DC Motors: basics

- DC motor contains stationary magnetic field provided by permanent or electromagnet with field strength B.
- Motor contains a rotating circuit called the armature (the "coils") through which a current $i_a(t)$ flows.
- Current flows perpendicular through the magnetic field and feels a force $F = Bli_a(t)$ acting on it.
- Parameter l is the length of the conductor the current is flowing through within the field.

- Associated with the armature is a resistance R_a, and an inductance L_a.

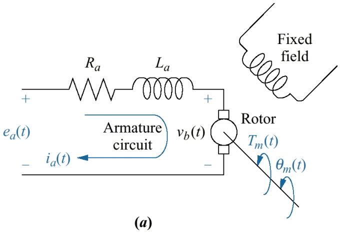


Figure 2.35.

Back EMF

- When a conductor moves at right angles to a magnetic field, it generates a voltage at terminals of the conductor (think generator).
- Since armature is rotating in a magnetic field, it produces a voltage proportional to its velocity.

$$v_b(t) = K_b \frac{d\theta_m(t)}{dt}$$

- We refer to $v_b(t)$ as the back electromotive force (back emf), and K_b as the *back emf constant.*

- Taking Laplace transform we get:

$$V_b(s) = K_b s \theta_m(s)$$

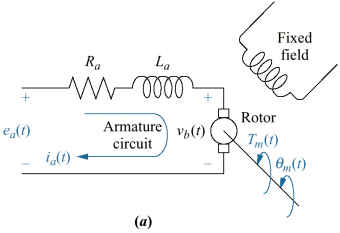


Figure 2.35.

Analyzing Armature Circuit

- We wish to find the transfer function: $\frac{\theta_m(s)}{E_a(s)}$
- Applying Kirchoff's voltage law to armature circuit gives:

$$R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s)$$

- The torque developed by motor is:

$$T_m(s) = K_t I_a(s) \;\Rightarrow I_a(s) = \frac{1}{K_t} T_m(s)$$

- Substituting in for $I_a(s)$ and $V_b(s)$ gives:

$$\frac{(R_a + L_a s)T_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$$

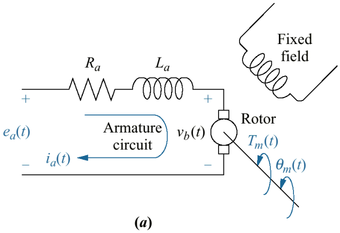


Figure 2.35.

Analyzing Armature Circuit - II

- Figure shows a typical mechanical loading of a motor.
- Variable J_m is the moment of inertia of armature plus that of the load reflected to the armature.
- Variable D_m is the viscous damping of the armature plus that of the load reflected to the armature.
- This gives the equation of motion:

$$T_m(s) = (J_m s^2 + D_m s)\theta_m(s)$$

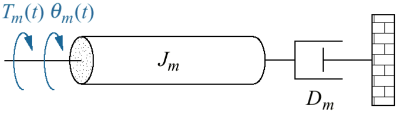


Figure 2.36.

DC Motor Transfer function

- We can now substitute into our previously derived equation for $E_a(s)$:

$$\frac{(R_a + L_a s)(J_m s^2 + D_m s)\theta_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$$

- and noting that typically $(R_a + L_a s) \approx R_a$ gives:

$$[\frac{R_a}{K_t}(J_m s + D_m) + K_b] s\theta_m(s) = E_a(s)$$

- This gives the transfer function:

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{K_t}{R_a J_m}}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]} = \frac{K}{s(s+\alpha)}$$

Deriving J_m and D_m

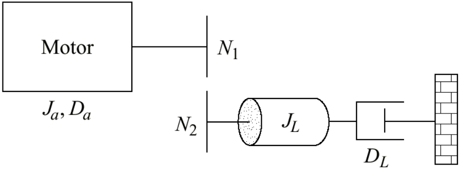


Figure 2.37.

- Figure shows typical usage of motor.
- Variable J_a is the moment of inertia of armature, and D_a is the viscous damping of the armature.
- Variable J_L is the load moment of inertia, and D_L is the viscous damping on the load.
- Reflecting onto the armature gives:

$$J_m = J_a + \left(\frac{N_1}{N_2}\right)^2 J_L \text{ and } D_m = D_a + \left(\frac{N_1}{N_2}\right)^2 D_L$$

Deriving Electrical Constants

- A *dynamometer* measues the torque and speed of a motor under a constant applied voltage.

- Previously, we derived:

$$\frac{R_a}{K_t}T_m(s) + K_b s\theta_m(s) = E_a(s)$$

- Inverse Laplace transform gives:

$$\frac{R_a}{K_t}T_m(t) + K_b\omega_m(t) = e_a(t)$$

- As voltage is constant, so is torque and velocity. Solving for T_m gives

$$T_m = -\frac{K_b K_t}{R_a}\omega_m + \frac{K_t}{R_a}e_a$$

Deriving Electrical Constants - II

$$T_m = -\frac{K_b K_t}{R_a}\omega_m + \frac{K_t}{R_a}e_a$$

- When $\omega_m = 0$ we get the *stall torque* given by:

$$T_{stall} = \frac{K_t}{R_a}e_a$$

- When $T_m = 0$ we get the angular velo
speed given by: $\omega_{no-load} = \frac{e_a}{K_b}$

- We thus have:

$$\frac{K_t}{R_a} = \frac{T_{stall}}{e_a} \text{ and } K_b = \frac{e_a}{\omega_{no-load}}$$

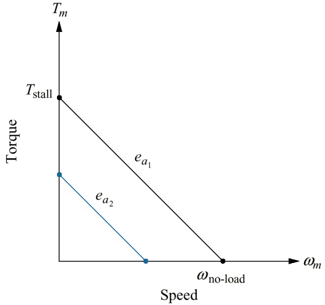


Figure 2.38.

Linear Systems

- So far, we have developed models for systems that can be approximately represented by *linear, time-invariant differential* equations.

- In developing these models, we assumed they were linear.

- A linear system must have two properties:

  Superposition: means that the output response to a sum of inputs is equal to the sum of the output response of each individual input that makes up the "sum of inputs."

  ie. if input $r_1(t)$ generates output response $c_1(t)$, and input $r_2(t)$ generates output response $c_2(t)$, the the output response to input $r_1(t) + r_2(t)$ will be $c_1(t) + c_2(t)$.

Linear Systems -II

Homogeneity: means that when the input is multipled by a scalar, the result is a response multiplied by same scalar

ie. if input $r_1(t)$ generates output response $c_1(t)$, then input $Ar_1(t)$ will result in output $Ac_1(t)$. Below (a) is linear, (b) is not.

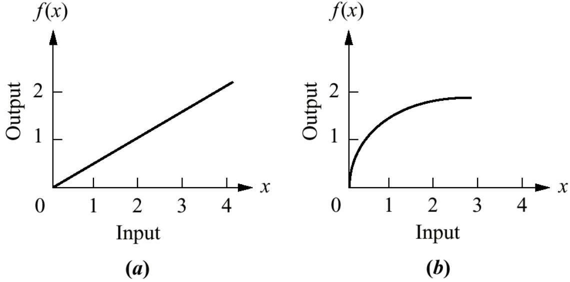


Figure 2.45.

Nonlinear Systems

- Nonlinear systems are systems that are not linear.
- Some examples are:
  - Op amps are linear over a given range, but exhibits saturation at high input voltages.
  - A motor exhibits a deadzone where it does not respond to low input voltages because of friction.
  - Backlash occurs when a pair of gears do not fit tightly. This is when the input gear moves through a small angle before the output gear starts to move.

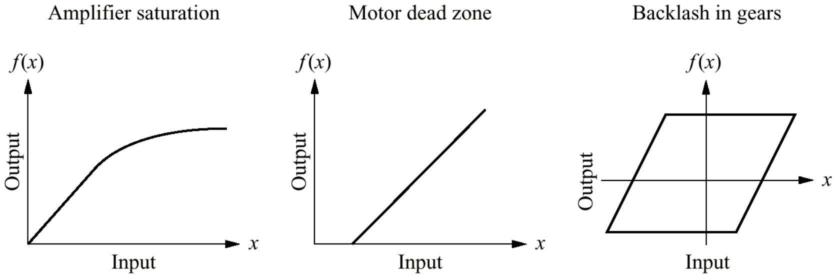


Figure 2.46.

Nonlinear Systems - II

- Sometimes we can make a linear approximation to a nonlinear system.
- For example, if system is linear for a portion, the range of input values about the point is small, and we can shift this portion to the origin, we can establish a linear relationship.
- Op amps are an example of this. We make sure our input values stay within the linear range.

Linearization

- If a system contains any nonlinear components, we must linearize the system before we can apply our linear systems modelling methods.

- We need to find linear approximations in order to be able to find transfer functions.

- To linearize a system, we do the following:
  1. Identify the nonlinear components and write nonlinear differential equations for them.
  2. We then choose a small range of input values over which the system behaves approximately linear. Refer to this range as a *small-signal input*.

     This range is centered around a steady state solution, called *equilibrium*, where the small-signal input is equal to zero.
  3. We then write a linear differential expression for this range, and then apply Laplace transforms and form our transfer function as normal.

Linearizing a Function

- Assume we have a nonlinear system that is operating about point A: $[x_o, f(x_o)]$.

- For small changes around A, we can calculate the approximate value of $f(x)$ using the slope of the curve at x_o.

- If slope at point A is m_a we can determine the change in $f(x)$, called $\delta f(x)$, for very small changes in x, called δx.

$$\delta f(x) = [f(x) - f(x_o)] \approx m_a(x - x_o) = m_a \delta x$$

- This gives us:

$$\begin{aligned} f(x) &\approx f(x_o) + m_a(x - x_o) \\ &\approx f(x_o) + m_a \delta x \end{aligned}$$

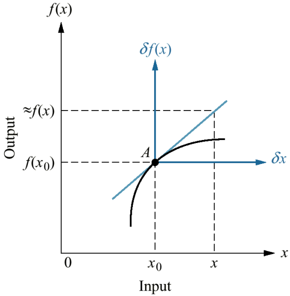


Figure 2.47.

Linearizing a Function - eg.

- Linearize $f(x) = 5\cos x$ about $\pi/2$.

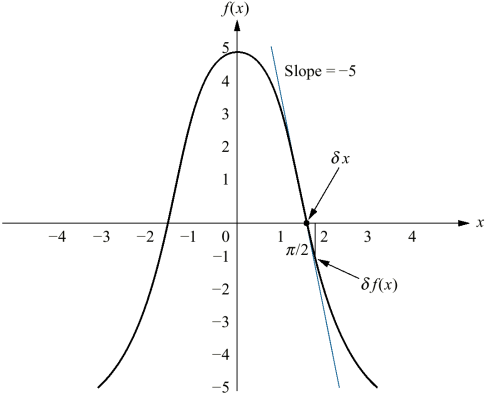


Figure 2.48.

To get slope of eqn at $x_o = \pi/2$ we take the derivative of $f(x)$:

$$f'(x) = \frac{df}{dx}(x) = \frac{d}{dx}(5\cos x) = -5\sin x$$

Linearizing a Function - eg. - II

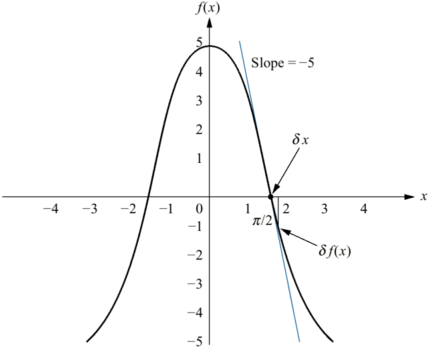


Figure 2.48.

Thus: $f'(\pi/2) = -5\sin(\pi/2) = -5 = m_a$

Also: $f(x_o) = f(\pi/2) = 5\cos(\pi/2) = 0$

We thus have:

$$f(x) = f(x_o) + m_a\delta x = 0 - 5\delta x = -5\delta x = -5(x - x_o)$$

Formalizing the Linearization Method.

- Using the Taylor series expansion, we can express the value of a function about point x_o using the equation:

$$f(x) = f(x_o) + \left.\frac{df}{dx}\right|_{x=x_o} \frac{(x - x_o)}{1!} + \left.\frac{d^2 f}{dx^2}\right|_{x=x_o} \frac{(x - x_o)^2}{2!} + \ldots$$

- For small range around x_o, we can neglect higher order terms and we get:

$$f(x) - f(x_o) \approx \left.\frac{df}{dx}\right|_{x=x_o} (x - x_o) = m|_{x=x_o} (x - x_o)$$

Linearizing a Differential Eqn eg.

- Linearize the differential equation below about $x_o = \pi/4$:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \cos x = 0$$

First, rewrite equation in terms of δx. As $\delta x = x - x_o$, we get $x = \delta x + x_o = \delta x + \pi/4$ giving:

$$\frac{d^2(\delta x + \pi/4)}{dt^2} + 2\frac{d(\delta x + \pi/4)}{dt} + \cos(\delta x + \pi/4) = 0$$

As the derivative of a constant is zero, the equation simplifies to:

$$\frac{d^2\delta x}{dt^2} + 2\frac{d\delta x}{dt} + \cos(\delta x + \pi/4) = 0 \tag{7}$$

We now need to evaluate: $f(x) - f(x_o) \approx \left.\frac{df}{dx}\right|_{x=x_o} \delta x$

Linearizing a Differential Eqn eg. - II

We now need to evaluate:

$$f(x) - f(x_o) \approx \left.\frac{df}{dx}\right|_{x=x_o} \delta x \qquad (8)$$

Our function is: $f(x) = \cos(x) = \cos(\delta x + \pi/4)$

We have $f(x_o) = f(\pi/4) = \cos(\pi/4)$ and

$$\left.\frac{d(\cos x)}{dx}\right|_{x=\pi/4} = -\sin(\pi/4)$$

Subbing into 8 gives:

$$\cos(\delta x + \pi/4) - \cos(\pi/4) = -\sin(\pi/4)\delta x$$

Linearizing a Differential Eqn eg. - III

Solving for $f(x)$ gives:

$$\cos(\delta x + \pi/4) = \cos(\pi/4) - \sin(\pi/4)\delta x = 0.7071 - 0.7071\delta x$$

Subbing into 7 gives:

$$\frac{d^2\delta x}{dt^2} + 2\frac{d\delta x}{dt} - 0.7071\delta x = -0.7071$$

We now have a linear differential equation.

If we wished to solve for x, we could take the Laplace transform of both sides (taking δx as a function of x) assuming zero initial conditions, solve for $\delta x(s)$, do inverse Lapalace transform, and then sub in for $\delta x = x - x_o$.

Transfer Function - Nonlinear Electrical System eg.

- Find transfer function $\frac{V_L(s)}{V(s)}$, where voltage current relationship for resistor is $i_r = 2e^{0.1v_r}$, and $v(t)$ is a small-signal voltage source.

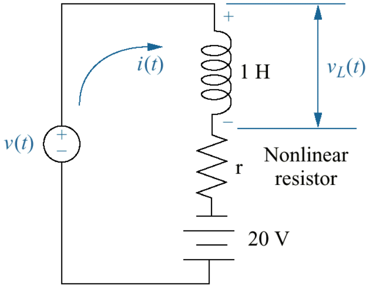


Figure 2.49.