See also problem

Problemè 1

In Lab 4, We used a PD compensator to control our ball and beam apparatus. The transfer function of our PD compensator was as follows:

$$G_C(s) = K_D s + K_P$$

However, we did not use the compensator in this form. The transfer function we used in lab was as follows:

$$G_C(s) = K_C(s + z)$$

> **Question**
>
> Solve for K_C and z in terms of K_P and K_D.

Given

$$\begin{aligned} G_C(s) &= K_C(s + z) \\ G_C(s) &= K_D s + K_P \end{aligned}$$

Or it can be written as:

$$(K_C - K_D)s + K_C z - K_P = 0$$

To solve for the characteristic equation, we can set the coefficients of s and the constant term to zero:

$$\begin{aligned} K_C - K_D &= 0 \\ K_C z - K_P &= 0 \end{aligned}$$

Thus, we can solve for K_C and z as follows:

$$\begin{aligned} K_C &= K_D \\ z &= \frac{K_P}{K_C} = \frac{K_P}{K_D} \end{aligned}$$

Problemè 2

Given that the transfer function of our Ball and Beam plant used in the previous lab is as follows:

$$G(s) = \frac{0.419}{s^2}$$

And given that the controller is applied to the plant in cascade configuration, find:

> ⓘ 2.a
>
> Static error constant for position (position constant)

This is a Type-2 system, thus position constant $K_p = \infty$

> ⓘ 2.b
>
> Static error constant for velocity (velocity constant)

Velocity constant $K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{0.419s}{s^2} = \infty$

> ⓘ 2.c
>
> Static error constant for acceleration (acceleration constant)

Acceleration constant $K_a = \lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} \frac{0.419s^2}{s^2} = 0.419$

> ⓘ 2.d
>
> Steady-state error for a step input $u(t)$

For a step input $R(s) = \frac{1}{s}$, the steady-state error is given by:

$$e_{ss} = \lim_{s \to 0} \frac{R(s)}{1 + K_p G(s)} = \lim_{s \to 0} \frac{1/s}{1 + \infty} = 0$$

2.e

Steady-state error for a ramp input $tu(t)$

For a ramp input $R(s) = \frac{1}{s^2}$, the steady-state error is given by:

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + K_v G(s)} = \lim_{s \to 0} \frac{s/s^2}{1 + 0.419} = \frac{1}{0.419} \approx 2.39$$

2.f

Steady-state error for a parabolic input $t^2 u(t)$

For a parabolic input $R(s) = \frac{1}{s^3}$, the steady-state error is given by:

$$e_{ss} = \lim_{s \to 0} \frac{s^2 R(s)}{1 + K_a G(s)} = \lim_{s \to 0} \frac{s^2/s^3}{1 + 0.419} = \frac{1}{0.419} \approx 2.39$$

Problemè 3

We will be augmenting our controller to include an integrator. The transfer function of our new PID compensator will be as follows;

$$G_C(s) = K_D s + K_P + \frac{K_I}{s}$$

Given that the transfer function for our plant has not changed, and given that this controller is also applied to the plant in cascade configuration.

The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G(s)G_C(s)}{1 + G(s)G_C(s)} = \frac{\frac{0.419}{s^2} * (K_D s + K_P + \frac{K_I}{s})}{1 + \frac{0.419}{s^2} * (K_D s + K_P + \frac{K_I}{s})} = \frac{0.419 * (K_D s + K_P + \frac{K_I}{s})}{s^2 + 0.419 * (K_D s + K_P + \frac{K_I}{s})}$$

3.a

Static error constant for position (position constant)

$$K_P = \lim_{s \to 0} G_C(s)G(s) = \lim_{s \to 0} \frac{0.419}{s^2}(K_D s + K_P + \frac{K_I}{s}) = \infty$$

3.b

Static error constant for velocity (velocity constant)

$$K_V = \lim_{s \to 0} sG_C(s)G(s) = \lim_{s \to 0} s\frac{0.419}{s^2}(K_D s + K_P + \frac{K_I}{s}) = 0.419K_D + \frac{0.419K_P}{s} + \frac{0.419K_I}{s^2} = 0.41$$

3.c

Static error constant for acceleration (acceleration constant)

$$K_A = \lim_{s \to 0} s^2 G_C(s)G(s) = \lim_{s \to 0} s^2\frac{0.419}{s^2}(K_D s + K_P + \frac{K_I}{s}) = 0.419K_P$$

3.d

Steady-state error for a step input $u(t)$

For a step input $R(s) = \frac{1}{s}$, the steady-state error is given by:

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 - \frac{C(s)}{R(s)}} = \lim_{s \to 0} s\frac{1/s}{1 - \frac{C(s)}{R(s)}} = \frac{1}{1 + K_P} = 0$$

3.e

Steady-state error for a ramp input $tu(t)$

For a ramp input $R(s) = \frac{1}{s^2}$, the steady-state error is given by:

$$e_{ss} = \lim_{s \to 0} \frac{s^2 R(s)}{1 - \frac{C(s)}{R(s)}} = \lim_{s \to 0} \frac{s^2/s^2}{1 - \frac{C(s)}{R(s)}} = \frac{1}{K_V} = \frac{1}{0.419K_I}$$

3.f

Steady-state error for a parabolic input $t^2u(t)$

For a parabolic input $R(s) = \frac{1}{s^3}$, the steady-state error is given by:

$$e_{ss} = \lim_{s \to 0} \frac{s^3 R(s)}{1 - \frac{C(s)}{R(s)}} = \lim_{s \to 0} \frac{s^3/s^3}{1 - \frac{C(s)}{R(s)}} = \frac{1}{K_A} = \frac{1}{0.419K_P}$$

Problemè 4

Ideally you want your controller design to reject a step disturbance input at $D(s)$. This means that in the steady state for $D(s) = \frac{1}{s}$, the output $Y(s)$ is unchanged.

ⓘ 4.a

Ignoring the input $R(s)$, what is the transfer function $\frac{E(s)}{D(s)}$ in terms of $G_1(s)$ and $G_2(s)$?

To find the transfer function $\frac{E(s)}{D(s)}$, then the transfer function $\frac{E(s)}{D(s)}$ is given by:

$$\frac{E(s)}{D(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)}$$

ⓘ 4.b

For $G_1(s) = K_C(s+z)$ and $G_2(s) = \frac{0.419}{s^2}$ what is the steady state error resulting from step inputs $R(s) = \frac{A}{s}$ and $D(s) = \frac{B}{s}$

The steady-state error to step input $R(s) = \frac{A}{s}$ is given by:

$$e_{ss}(R) = \lim_{s \to 0} \frac{A}{s} \left(\frac{1}{1 + L(s)}\right)$$

with $L(s) = G_1(s)G_2(s) = \frac{0.419K_C(s+z)}{s^2}$

$$e_{ss}(R) = \lim_{s \to 0} \frac{A}{s} \left(\frac{1}{1 + \frac{0.419K_C(s+z)}{s^2}}\right) = \frac{A}{0.419K_C}$$

The steady-state error to step input $D(s) = \frac{B}{s}$ is given by: $e_{ss}(D) = \frac{B}{0.419K_C}$

Thus, the total steady-state error is $e_{ss} = e_{ss}(R) + e_{ss}(D) = \frac{A+B}{0.419K_C}$

? 4.c

For $G_1(s) = K_D s + K_P + \frac{K_I}{s}$ and $G_2(s) = \frac{0.419}{s^2}$ what is the steady state error resulting from step inputs $R(s) = \frac{A}{s}$ and $D(s) = \frac{B}{s}$

$$L(s) = G_1(s)G_2(s) = \frac{0.419(K_D s + K_P + \frac{K_I}{s})}{s^2} = \frac{0.419K_D s^2 + 0.419K_P s + 0.419K_I}{s^3}$$

The steady-state error to step input $R(s) = \frac{A}{s}$ is zero:

$$e_{ss}(R) = \lim_{s\to 0} \frac{A}{s}(\frac{1}{1+L(s)}) = \lim_{s\to 0} \frac{A}{s}(\frac{1}{1+\frac{0.419K_D s^2+0.419K_P s+0.419K_I}{s^3}}) = 0$$