Introduction

- In this Section, we examine ways to determine if a system is stable.
- Of all design criteria, stability is most important.
- If system is unstable, then transient response and steady-state error are irrelevant.
- We will now examine a few stability definitions for linear, time-invariant systems.

Stability and Natural Response

- The total response of a system is

$$c(t) = c_{forced}(t) + c_{natural}(t)$$

1. A system is stable if natural response tends to zero as $t \to \infty$.
2. A system is unstable if natural response grows unbounded as $t \to \infty$.
3. A system is marginally stable if natural response neither decays or grows (eg. stays constant or oscillates with fixed amplitude) as $t \to \infty$.

- Definition of "stable" implies that as $t \to \infty$, only the forced response remains.

Bounded-input, Bounded-output (BIBO) Stability

- The BIBO definition is in terms of the total response, so you don't need to isolate the natural response first.

1. A system is stable if *every* bounded input produces a bounded output.

2. A system is unstable if *any* bounded input produces an unbounded output.

Stability and Poles

- In order to easily determine if a system is stable, we can examine the poles of the closed-loop system.

1. A system is stable if all the poles are strictly on the left hand side of the complex plane.
2. A system is unstable if any pole is in the right hand side of the complex plane or the system has poles on the imaginary axis that are of multiplicity > 1.
3. A system is marginally stable if no pole is on the right hand side, and its poles on the imaginary axis are of multiplicity one.

ie. $\dfrac{1}{(s^2 + \omega^2)}$ marginally stable, but $\dfrac{1}{(s^2 + \omega^2)^2}$ is not.

Stability and Poles - II

- Poles on the imaginary axis of multiplicity greater than one have time responses of the form $At^n \cos(\omega t + \phi)$ which tend to infinity as $t \to \infty$.

- This implies systems with poles on the imaginary axis of multiplicty one will be unstable by the BIBO definition as a sinusoid input at same frequency (ω) will result in a total response with imaginary poles of multiplicty two!

Stability and Poles - III

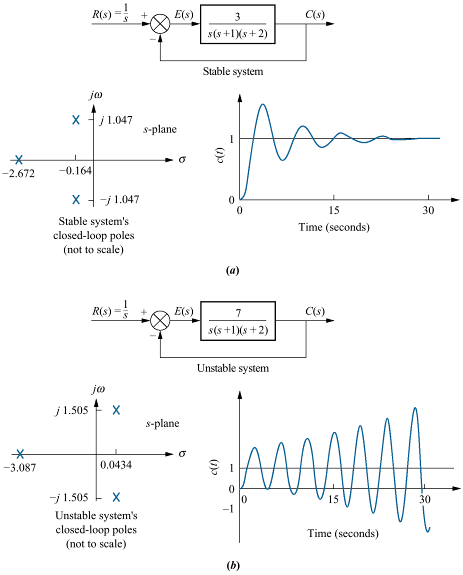


Figure 6.1.

Stability Summary

Table: Stability Comparison

Real Part of Poles	Natural Response	BIBO
All poles < 0	stable	stable
Any pole > 0 or imaginary poles of multiplicity > 1	unstable	unstable
Poles ≤ 0 and imaginary poles of multiplicity one	marginally stable	unstable

Closed-loop Systems

- If the poles of the original system are not as desired, we can use feedback control to move the poles.
- In Fig. 6.2(a) below, we can easily see the poles of original system, but we don't know the poles of closed-loop system without factoring.
- Would like an easy way to tell if the closed-loop system is stable without having to factor it.

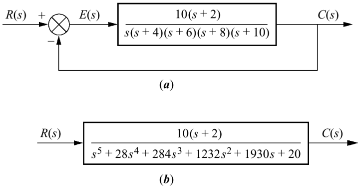


Figure 6.2.

Necessary Stability Condition

- A *necessary* condition for a polynomial to have all roots in the open left hand plane is to have all coefficients of the polynomial to be present and to have the same sign.
- However, this is not a *sufficient* condition.
- A *sufficent* condition that a system is *unstable* is that all coefficients do not have the same sign.
- If some coefficents are missing, system MAY be unstable, or at best, marginally stable.
- If all coefficients are same sign and present, system could be stable or unstable.

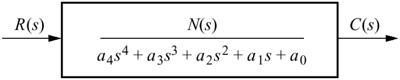


Figure 6.3.

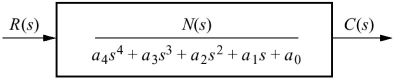
Routh-Hurwitz Criterion

- This method will give us stability info without having to find poles of closed-loop system.
- Will tell us:
  - How many poles in left half-plane.
  - How many poles in right half-plane.
  - How many poles on imaginary axis.
- Method called Routh-Hurwitz criterion for stability.
- To apply method we need to:
  1. Construct a table of data called a *Routh table*.
  2. Interpret the table to determine the above classifications.

Creating a Basic Routh Table

- The Routh-Hurwitz criterion focusses on the coefficients of the denominator of the transfer function.

1. Label the rows of the table with powers of s, starting from the highest power down to s^0.
2. List coefficients across top row, starting with coefficent of the highest power of s, and then every other coefficient.
3. List remaining coefficents in second row, starting with coefficient of second highest power.



s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Figure 6.3 and Table 6.1.

Creating a Basic Routh Table - II

- To fill in remaining rows as follows:

1. Each entry is a negative determinant of entries from the previous two rows.
2. Each determinant is divided by the entry in the first column of the row above.
3. Left column of determinant is first column of the previous two rows.
4. Right column contains elements of the column above and directly to the right of the current location.
5. If no column to right, use zeros.

Table 6.2

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Interpreting a Basic Routh Table

- Basic Routh table applies to systems with poles in open left or right hand plane, but no imaginary poles.
- The Routh-Hurwitz criterion states that the number of poles in the right half plane is equal to the number of sign changes in the first coefficient column of the table.
- A system is stable if there are no sign changes in the first column.

Basic Routh Table eg.

- Apply the Routh-Hurwitz criterion to the system below to determine stability.
- One may multiply any row by a positive constant without changing the values of the signs of the rows below.
- You MUST not multiple a row by a negative constant.

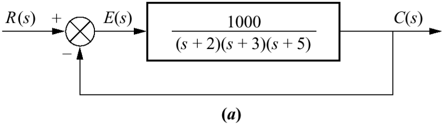


Figure 6.4.

Case I: Zero Only in First Column - ϵ

- Consider system with closed-loop transfer function:

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3} \quad (1)$$

- Replace zero entry in first column by an ϵ (very small number), then complete table.
- Examine table by allowing ϵ to approach zero from the positive and negative side.

s^5	1	3	5
s^4	2	6	3
s^3	~~0~~ ϵ	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

Label	First Column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	~~0~~ ϵ	+	−
s^2	$\frac{6\epsilon - 7}{\epsilon}$	−	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

Tables 6.4 and 6.5.

Case I: Zero Only in First Column - Reciprocal

- A polynomial whose roots are the reciprocal of the original polynomial, has poles with same distributions (ie. # in left side, right side, imaginary).
- This new polynomial might not have a zero in the first column.
- Can find this polynomial simply by reversing order of coefficients.

$T(s) = \frac{10}{s^5+2s^4+3s^3+6s^2+5s+3} \rightarrow D(s) = \frac{10}{1+2s^1+3s^2+6s^3+5s^4+3s^5}$

$D(s) = \frac{10}{3s^5+5s^4+6s^3+3s^2+2s^1+1}$

s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	−1.75		
s^0	1		

Table 6.6.

Case II: Row of Zeros

- Consider closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56} \qquad (2)$$

- When evaluating row s^3, we find all entries to be zero.
- To proceed, form polynomial using coefficients of row above the zero row.
- Start with power of row above the zero row, and then skip every other power of s.
- This gives us:

$$P(s) = s^4 + 6s^2 + 8 \qquad (3)$$

- Next, differentiate with respect to s

$$\frac{dP(s)}{ds} = 4s^3 + 12s^1 + 0 \qquad (4)$$

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

Table 6.7

Case II: Row of Zeros - II

- Replace row of zeros with coefficients of polynomial from equation 4, and continue.

Table 6.7

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

Why Row of Zeros?

- We get a row of zeros when original polynomial has a purely even or odd polynomial as a factor.
- A purely even polynomial is one where all powers of s are even.
- An even polynomial only has roots that are symmetrical about origin.
- As $j\omega$ roots are symmetric across origin, they can only occur when we have a row of zeros.

- In Routh table, the row above the row of zeros contains the even/odd polynomial that is a factor of the original polynomial.

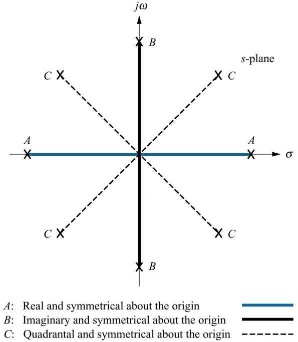


Figure 6.5

Why Row of Zeros? - II

- Also, Everything from row containing even polynomial onwards is a test of *only* the even polynomial.
- Returning to our example with $T(s) = \frac{10}{s^5+7s^4+6s^3+42s^2+8s+56}$, we see that it had even polynomial $P(s) = s^4 + 6s^2 + 8$ as a factor.
- Rows s^4 to s^0 thus give information only about $P(s)$.
- As there are no sign changes, we thus have 4 imaginary poles.
- As $P(s)$ is not perfect fourth order square polynomial, the imaginary poles are of multiplicity 1.
- There are no sign changes from rows s^5 to s^4, so our last pole is in left hand side.

- System is thus marginally stable.

Table 6.7

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

Stability Design via Routh-Hurwitz

- Changes in the gain of systems like the one below, can result in changes of the closed-loop pole locations.
- In the next example, we can use the Routh-Hurwitz criterion to show that gain changes can move stable poles from the right-hand plane, to the imaginary axis, to the left-hand plane.

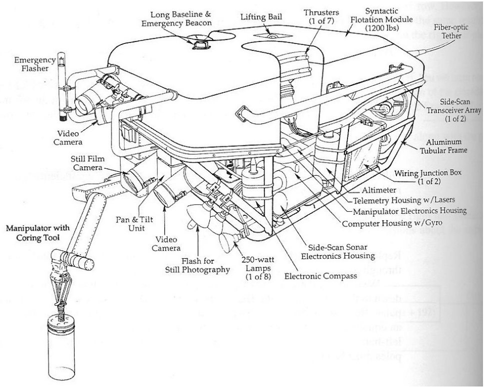


Figure 6.9.

Stability Design via Routh-Hurwitz eg.

- Find range of K (gain) that will make the system stable, marginally stable, and unstable.

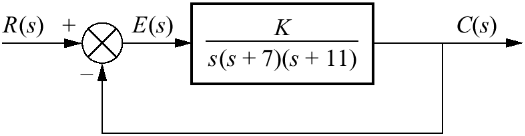


Figure 6.10.

Stability in State Space

- For a state space system, we are given the state and output equations below

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \qquad \textit{state equations}$$

$$\underline{y} = \underline{C}\,\underline{x} + \underline{D}\,\underline{u} \qquad \textit{output equations}$$

- If we have a single input, single output system, we can use the equation below to find the corresponding transfer function. :

$$G(s) = \frac{Y(s)}{U(s)} = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} + \underline{D} \tag{5}$$

  If its has multiple inputs or outputs the we get $\underline{G}(S)$, a TF matrix.

- From linear algebra we know:

$$[s\underline{I} - \underline{A}]^{-1} = \frac{\mathrm{adj}([s\underline{I} - \underline{A}])}{\det([s\underline{I} - \underline{A}])} \tag{6}$$

Stability in State Space - II

- Substituting equation 6 into equation 5, we get

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\underline{C}\,\text{adj}([s\underline{I} - \underline{A}])\,\underline{B}}{\text{det}([s\underline{I} - \underline{A}])} + \underline{D} = \frac{N(s)}{D(s)} \tag{7}$$

$$= \frac{\underline{C}\,\text{adj}([s\underline{I} - \underline{A}])\,\underline{B} + \det([s\underline{I} - \underline{A}])D}{\text{det}([s\underline{I} - \underline{A}])} \tag{8}$$

- We thus have:

$$\text{det}([s\underline{I} - \underline{A}]) = D(s) \tag{9}$$

- We define the roots of the equation $\text{det}([s\underline{I} - \underline{A}]) = 0$ to be the eigenvalues of matrix $\underline{A}$.
- To determine if a state space system is stable, we determine the eigenvalues of matrix $\underline{A}$, and then determine their location in the s-plane, using the same rules for stability as for the poles of a transfer function.