Introduction

- Frequency response design methods allow us to place the dominant second-order pair of poles.
- We then hope that the higher-order poles won't invalidate the approximation.
- We want to be able to specify the location of all n poles.
- We need n adjustable parameters to place n unknown values.
- A single gain and a compensator pole and zero are typically not enough.

Introduction - II

- State space methods solve this by:
  1. introducing into the system other adjustable parameters
  2. providing techniques to determine values for these parameters that will correctly place the poles.
- A disadvantage of state space methods is that it doesn't allow the placement of closed loop zeros which can affect transient response.
- Also, a state space design may be quite sensitive to changes in parameters.

Control Design

- An n^{th} order feedback control system has a n^{th} order closed-loop characteristic equation given by

$$\det(s\mathbf{I} - \mathbf{A}') = s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_o = 0$$

  where $\mathbf{A}'$ is the closed loop system matrix.
- The characteristic equation contains n coefficients that determine the system's n poles (eigenvalues).
- Our goal is to introduce n new adjustable parameters and relate them to these coefficients.

Topology for Pole Placement

- Consider a plant represented as

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}u \quad (1)$$

$$y = \mathbf{C}\,\mathbf{x} \quad (2)$$

- Typically, the output y is fed back.
- Instead, we feed back each state variable with its own gain, k_i.

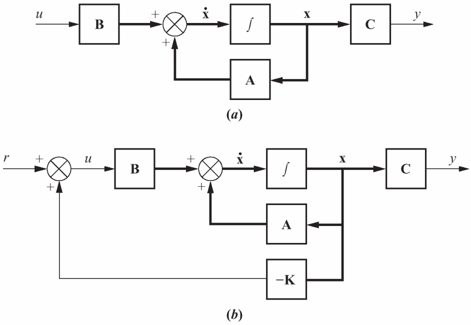


Figure 12.2.

Topology for Pole Placement - II

- We represent the gains by feedback vector $-\mathbf{K}$.
- This gives a closed-loop system as represented as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}u = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,(-\mathbf{K}\,\mathbf{x} + r) = (\mathbf{A} - \mathbf{B}\,\mathbf{K})\,\mathbf{x} + \mathbf{B}\,r \quad (3)$$

$$y = \mathbf{C}\,\mathbf{x} \quad (4)$$

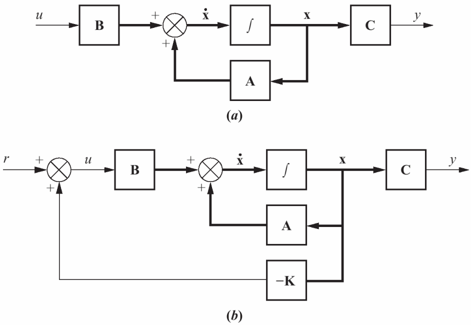


Figure 12.2.

Signal Flow Graph

- Signal flow graphs are an alternative to block diagrams.
- They consist of nodes which represent signals, and branches that represent systems (the blocks of block diagrams).
- Value of a node is the sum of the signals entering it.
- To subtract an incoming signal, label the branch as negative.
- For example:

$$V(S) = R_1(s)G_1(s) - R_2(s)G_2(s) + R_3(s)G_3(s)$$

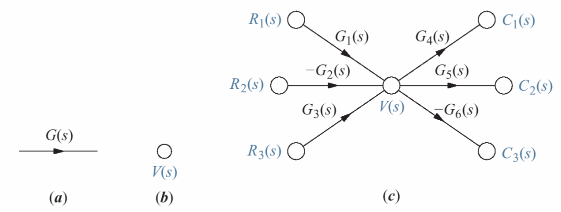


Figure 5.17.

Signal Flow Graph e.g.

- Convert the block diagram below to a signal flow graph.
- The steps are:
  1. First, draw all the signal nodes of the system.
  2. Add the branches to connect the nodes.
  3. Simplify the diagram by eliminating nodes with a single entry and exit point.

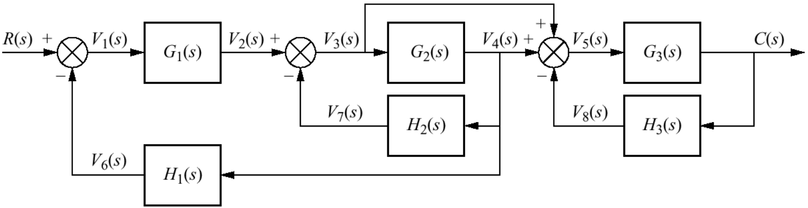


Figure 5.11.

Signal Flow Graph e.g. - II

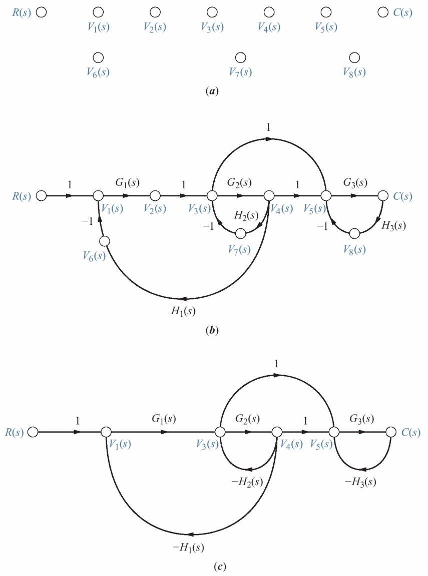


Figure 5.19.

Phase Variable (Controller Cannonical) Form

- System with transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_o}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0}$$

- gives differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u$$

- Our state variables and first-order differential equations are as follows:

$$x_1 = y,\ x_2 = \frac{dy}{dt},\ x_3 = \frac{d^2y}{dt^2}, \cdots, x_n = \frac{d^{n-1}y}{dt^{n-1}}$$

$$\dot{x_1} = x_2,\ \dot{x_2} = x_3,\ \dot{x_3} = x_4, \cdots, \dot{x}_{n-1} = x_n$$

$$\dot{x_n} = -a_o x_1 - a_1 x_2 \cdots - a_{n-1}x_n + b_0 u$$

Phase Variable (Controller Cannonical) Form - II

- Putting the state equations in matrix form gives:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u$$

- Using that our ouput $y(t)$ equals x_1, gives:

$$y = [1 \quad 0 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_0 u$$

Phase Variable Form - Zeros

- As we saw before, if numerator is not a constant, the numerator defines the output equation.
- The output below is thus:

$$\begin{aligned} Y(s) &= (s^2 + 7s + 2)X_1(s) = s^2 X_1(s) + 7sX_1(s) + 2X_1(s) \\ &= X_3(s) + 7X_2(s) + 2X_1(s) \end{aligned}$$

- In the time domain we thus have:

$$y = 2x_1 + 7x_2 + x_3$$

- We thus have ouput matrix $C = [2 \;\; 7 \;\; 1]$.

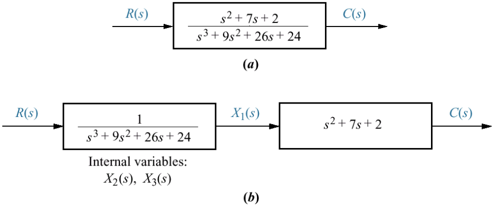
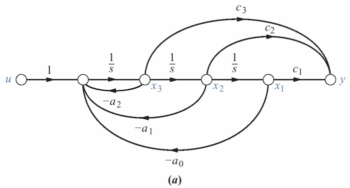


Figure 3.12.

State Feedback Example

- Below is an example of plant in phase-variable form with state feedback added.



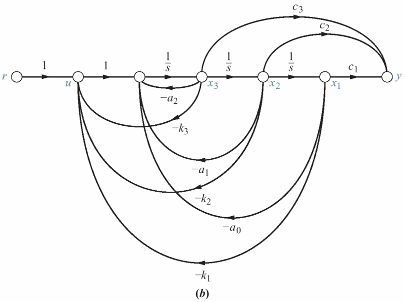


Figure 12.3.

Pole Placement with Phase-Variable Form

- To apply the pole placment approach with plants in phase-variable form, we follow the following steps.
  1. Represent the plant in phase-variable form.
  2. Feed back each state variable via gain k_i.
  3. Find characteristic eqn for above system.
  4. Select desired closed-loop poles and corresponding characteristic equation.
  5. Equate coefficients of characteristic equations from last two steps, and solve for the k_i.

Pole Placement with Phase-Variable Form - II

- Phase-variable representation of plant is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

$$\mathbf{C} = [c_1 \quad c_2 \quad \cdots \quad c_n]$$

- Can show above system has characteristic equation:

$$s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0$$

- We then feed back each state variable to input u giving:

$$u = -\mathbf{K}\,\mathbf{x}, \quad \text{where } \mathbf{K} = [k_1 \;\; k_2 \;\; \cdots k_n]$$

- For our closed-loop system, this gives system matrix:

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -(a_0 + k_1) & -(a_1 + k_2) & -(a_2 + k_3) & \cdots & -(a_{n-1} + k_n) \end{bmatrix}$$

Pole Placement with Phase-Variable Form - III

- Closed-loop system thus has characteristic equation:

$$\det(s\mathbf{I} - (\mathbf{A} - \mathbf{B}\,\mathbf{K})) = s^n + (a_{n-1} + k_n)s^{n-1} + (a_{n-2} + k_{n-1})s^{n-2} + \cdots + (a_1 + k_2)s + (a_0 + k_1) = 0 \quad (5)$$

- Assume that the desired closed-loop poles correspond to the characteristic equation:

$$s^n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \cdots + d_1 s + d_0 = 0 \quad (6)$$

- equating coefficients we get:

$$d_i = a_i + k_{i+1} \;\; \text{for} \;\; i = 0, 1, 2, \ldots n-1$$

- We thus have:

$$k_{i+1} = d_i - a_i$$

Pole Placement with Phase-Variable Form - e.g.

- Given plant below, design the phase-variable feedback gains to yield 9.5% overshoot and a settling time of 0.74 seconds.

$$G(s) = \frac{20(s+5)}{s(s+1)(s+4)} = \frac{20s+100}{s^3+5s^2+4s}$$

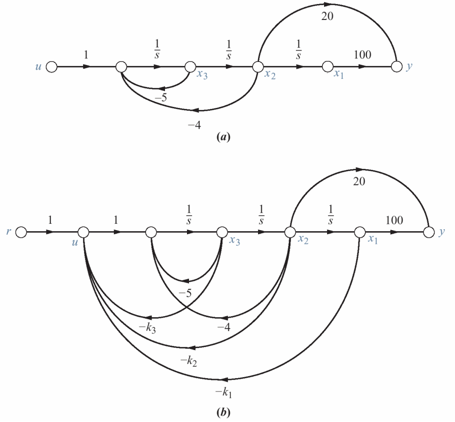


Figure 12.4.

Pole Placement with Phase-Variable Form - e.g. - II

- Start by determining location of dominant second-order poles to achieve desired transient response.
- Using methods from previous chapters, we find that needed poles are $s_{1,2} = -5.4 \pm j7.2$.
- As system is third order, we need to choose a location for third pole. We could:
  1. choose pole to be more than 5 times to the left of dominant second-order poles.
  2. choose pole to cancel zero
  3. optimize pole location to satisfy additional criteria.
- We should place pole at $s = -5$ to cancel the zero, but we will instead place pole at $s = -5.1$ to demonstrate why zero needs to be cancelled and the need for a final simulation.
- Desired characteristic equation is thus

$$(s + 5.4 + j7.2)(s + 5.4 - j7.2)(s + 5.1) = s^3 + 15.9s^2 + 136.08s + 413.1 \quad (7)$$

Pole Placement with Phase-Variable - e.g. - III

- From diagram and phase-varaiable form of system, we can derive the closed loop system as:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -(4+k_2) & -(5+k_3) \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [100 \quad 20 \quad 0]\mathbf{x}$$

- Our closed-loop system matrix is thus:

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -(4+k_2) & -(5+k_3) \end{bmatrix}$$

- Closed loop system's chracteristic equation is thus:

$$\det(s\mathbf{I} - (\mathbf{A} - \mathbf{B}\,\mathbf{K})) = s^3 + (5 + k_3)s^2 + (4 + k_2)s + k_1 = 0 \tag{8}$$

Pole Placement with Phase-Variable - e.g. - IV

- Comparing coefficients of Equations 7 and 8 gives:

$$5 + k_3 = 15.9; \quad 4 + k_2 = 136.08 \quad k_1 = 413.1$$

- We thus have:

$$k_1 = 413.1; \quad k_2 = 132.08; \quad k_3 = 10.9$$

- As the zeros of open-loop system are the same as the closed-loop system, our final system is thus:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -413.1 & -136.08 & -15.9 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [100 \quad 20 \quad 0]\mathbf{x}$$

- This gives a closed loop transfer function of:

$$\frac{20(s+5)}{s^3 + 15.9s^2 + 136.08s + 413.1}$$

Pole Placement with Phase-Variable - e.g. - V

- Simulation system gives 11.5% overshoot and 0.8 second settling time.

- Redesigning system with third pole at $s = -5$ gives correct result.

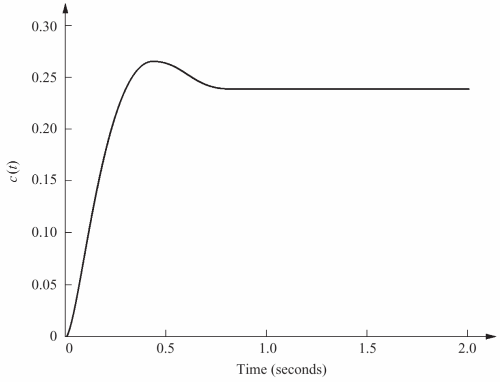


Figure 12.5.

Controllability

- It is not always possible to be able to place every pole in a system.
- In Figure (b) below, we can not use input u to control state x_1 as input u has no effect on this state.
- We say a system is (completely) controllable if we can find an input to a system that will take each state variable from a chosen initial state to a chosen final state. Otherwise system is uncontrollable.

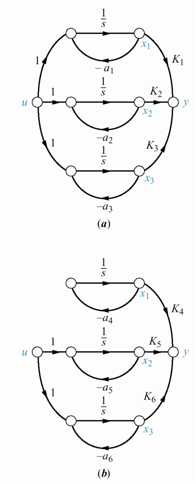


Figure: 12.6

Controllability - II

- For many systems, it is not obvious from inspection whether a system is controllable or not.
- For a system with state equation

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u},$$

  consider the so called controllability matrix, $\mathbf{C_M}$, below:

$$\mathbf{C_M} = [\mathbf{B}\ \ \mathbf{A}\,\mathbf{B}\ \ \mathbf{A}^2\,\mathbf{B}\ \ \cdots\ \ \mathbf{A}^{n-1}\,\mathbf{B}]$$

- It can be shown that if $\mathbf{C_M}$ is of rank n, then the system is controllable (see Ogata, K. *Modern Control Engineering*, 2d ed. Prentice Hall, Englewood Cliffs, NJ, 1990).
- The rank of a matrix is the maximum number of independent rows or columns.
- If determinant of a $n \times n$ matrix does not equal zero, then the matrix has rank n.

Controllability Example

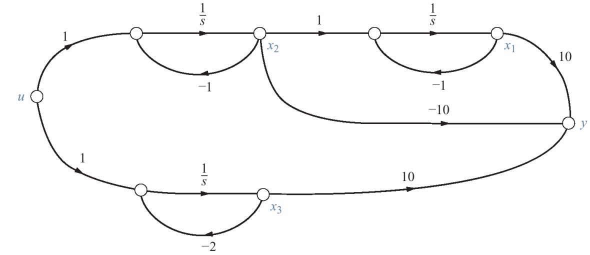


Figure 12.7.

Controllability Example

- Given state equation for system below

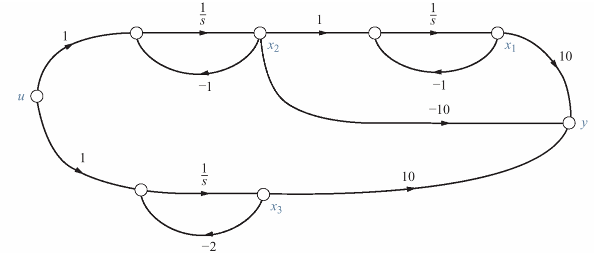


Figure 12.7.

- $$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

Controllability e.g.

- The controllability matrix is

$$\mathbf{C_M} = [\mathbf{B} \;\; \mathbf{A}\,\mathbf{B} \;\; \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{bmatrix}$$

- As $\det(\mathbf{C_M}) = -1$ (i.e. non zero), matrix $\mathbf{C_M}$ has rank 3 so system is controllable.

Alternate Approaches to Controller Design

- Pole placement is very straightforward when system in phase-variable form.
- For other forms, we can still evaluate the closed-loop and desire characteristic equation and compare coefficients, but the results typically lead to difficult calculations.
- An easier method is to transform the system into phase-variable form, place the poles, and then transform the result back into the original form.

Controller Design by Transform

- Assume plant below is NOT in phase-variable form

$$\dot{\mathbf{z}} = \mathbf{A}\,\mathbf{z} + \mathbf{B}\,u \qquad (9)$$
$$y = \mathbf{C}\,\mathbf{z}$$

- Corresponding controllability matrix is thus

$$\mathbf{C_{M_z}} = [\mathbf{B} \;\; \mathbf{A}\,\mathbf{B} \;\; \mathbf{A}^2\,\mathbf{B} \;\; \cdots \;\; \mathbf{A}^{n-1}\,\mathbf{B}] \qquad (10)$$

- We then assume that can convert the system into phase-variable form using the transformation

$$\mathbf{z} = \mathbf{P}\,\mathbf{x} \qquad (11)$$

- Substituting this into Equation 9, we get

$$\dot{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{A}\,\mathbf{P}\,\mathbf{x} + \mathbf{P}^{-1}\mathbf{B}\,u \qquad (12)$$
$$y = \mathbf{C}\,\mathbf{P}\,\mathbf{x}$$

Controller Design by Transform - II

- Corresponding controllability matrix is thus

$$\mathbf{C}_{\mathbf{M_x}} = [\mathbf{P}^{-1}\mathbf{B} \;\; (\mathbf{P}^{-1}\mathbf{A}\,\mathbf{P})\,(\mathbf{P}^{-1}\mathbf{B}) \;\; (\mathbf{P}^{-1}\mathbf{A}\,\mathbf{P})^2\,(\mathbf{P}^{-1}\mathbf{B}) \;\; \cdots$$
$$(\mathbf{P}^{-1}\mathbf{A}\,\mathbf{P})^{n-1}\,(\mathbf{P}^{-1}\mathbf{B})]$$
$$= [\mathbf{P}^{-1}\mathbf{B} \;\; (\mathbf{P}^{-1}\mathbf{A}\,\mathbf{P})\,(\mathbf{P}^{-1}\mathbf{B}) \;\; (\mathbf{P}^{-1}\mathbf{A}\,\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\,\mathbf{P})\,(\mathbf{P}^{-1}\mathbf{B})$$
$$\cdots \;\; (\mathbf{P}^{-1}\mathbf{A}\,\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\,\mathbf{P})\ldots(\mathbf{P}^{-1}\mathbf{A}\,\mathbf{P})(\mathbf{P}^{-1}\mathbf{B})]$$
$$= \mathbf{P}^{-1}[\mathbf{B} \;\; \mathbf{A}\,\mathbf{B} \;\; \mathbf{A}^2\,\mathbf{B} \;\; \cdots \;\; \mathbf{A}^{n-1}\,\mathbf{B}] = \mathbf{P}^{-1}\mathbf{C}_{\mathbf{M_z}}$$

- Solving for $\mathbf{P}$ gives

$$\mathbf{P} = \mathbf{C}_{\mathbf{M_z}}\,\mathbf{C}_{\mathbf{M_x}}{}^{-1} \tag{13}$$

Controller Design by Transform - III

- Once we have phase-variable form of system, we can design controller by setting $u = -\mathbf{K_x}\,\mathbf{x} + r$ giving

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{P}^{-1}\mathbf{A}\,\mathbf{P}\,\mathbf{x} - \mathbf{P}^{-1}\mathbf{B}\mathbf{K_x}\,\mathbf{x} + \mathbf{P}^{-1}\mathbf{B}\,r \\ &= (\mathbf{P}^{-1}\mathbf{A}\,\mathbf{P}\ - \mathbf{P}^{-1}\mathbf{B}\mathbf{K_x})\mathbf{x} + \mathbf{P}^{-1}\mathbf{B}\,r \\ y &= \mathbf{C}\,\mathbf{P}\,\mathbf{x}\end{aligned} \tag{14}$$

- We thus use system matrix $(\mathbf{P}^{-1}\mathbf{A}\,\mathbf{P}\ - \mathbf{P}^{-1}\mathbf{B}\mathbf{K_x})$ to construct our closed-loop characteristic equation, and solve for the elements of $\mathbf{K_x}$.
- We now use $\mathbf{x} = \mathbf{P}^{-1}\mathbf{z}$ to transform the above system back into the original system form giving us:

$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{A}\,\mathbf{z} - \mathbf{B}\mathbf{K_x}\,\mathbf{P}^{-1}\mathbf{z} + \mathbf{B}\,r \\ &= (\mathbf{A} - \mathbf{B}\mathbf{K_x}\,\mathbf{P}^{-1})\mathbf{z} + \mathbf{B}\,r\end{aligned} \tag{15}$$

- As standard form closed-loop system matrix is $(\mathbf{A} - \mathbf{B}\mathbf{K})$, we see that $\mathbf{K_z} = \mathbf{K_x}\,\mathbf{P}^{-1}$.

Controller Design by Transform - e.g.

- Design state-variable feedback controller that has 20.8% overshoot and settling time of 4 seconds for plant

$$G(s) = \frac{(s+4)}{(s+1)(s+2)(s+5)} = \frac{(s+4)}{s^3+8s^2+17s+10}$$

that is represented in cascade form (see Section 5.7 of text for more information about cascade form) below:

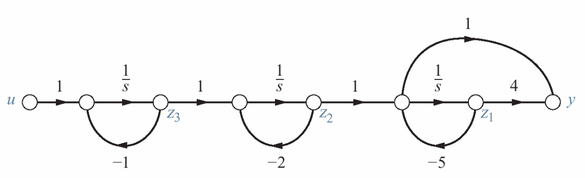


Figure 12.9.

Controller Design by Transform - e.g. II

- From the diagram, we can derive the system below

$$\dot{\mathbf{z}} = \mathbf{A_z}\,\mathbf{z} + \mathbf{B_z}\,u = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad (16)$$

$$y = \mathbf{C_z}\,\mathbf{z} = [-1 \;\; 1 \;\; 0]\mathbf{z}$$

- The corresponding controllability matrix is

$$\mathbf{C}_{\mathbf{M_z}} = [\mathbf{B_z} \;\; \mathbf{A_z}\,\mathbf{B_z} \;\; {\mathbf{A_z}}^2\,\mathbf{B_z}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix} \qquad (17)$$

- as $\det(\mathbf{C}_{\mathbf{M_z}}) = -1$, the system is controllable.

Controller Design by Transform - e.g. III

- Using characteristic equation ($\det(s\mathbf{I} - \mathbf{A_z})$) or denominator of $G(s)$, we can write out phase-variable form for system equations

$$\dot{\mathbf{x}} = \mathbf{A_x}\,\mathbf{x} + \mathbf{B_x}\,u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad (18)$$

- The corresponding controllability matrix is

$$\mathbf{C_{M_x}} = [\mathbf{B_x} \;\; \mathbf{A_x}\,\mathbf{B_x} \;\; \mathbf{A_x}^2\,\mathbf{B_x}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -8 \\ 1 & -8 & 47 \end{bmatrix} \quad (19)$$

- We thus have

$$\mathbf{P} = \mathbf{C_{M_z}}\,\mathbf{C_{M_x}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 10 & 7 & 1 \end{bmatrix} \quad (20)$$

Controller Design by Transform - e.g. IV

- We can now design our state-feedback gains (K_x) for phase-variable systems like before.
- To achieve system with desired specifications, we need our second-order system to be $s^2 + 2s + 5$.
- We place our third pole at $s = -4$ to cancel the zero.
- This gives desired characteristic equation

$$D(s) = (s+4)(s^2 + 2s + 5) = s^3 + 6s^2 + 13s + 20 = 0 \tag{21}$$

- The closed-loop system matrix is thus

$$\mathbf{A_X} - \mathbf{B_x K_x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(10 + k_{1_x}) & -(17 + k_{2_x}) & -(8 + k_{3_x}) \end{bmatrix} \tag{22}$$

Controller Design by Transform - e.g. V

- Corresponding characteristic equation is thus

$$\det(s\mathbf{I} - (\mathbf{A_x} - \mathbf{B_x}\,\mathbf{K_x})) = s^3 + (8 + k_{3_x})s^2 + (17 + k_{2_x})s + (10 + k_{1_x}) \quad (23)$$

- Comparing coefficients, we see that

$$\mathbf{K_x} = [k_{1_x} \;\; k_{2_x} \;\; k_{3_x}] = [10 \;\; -4 \;\; -2] \quad (24)$$

- Transforming the controller back to original system gives

$$\mathbf{K_z} = \mathbf{K_x}\mathbf{P}^{-1} = [-20 \;\; 10 \;\; -2] \quad (25)$$

- Combining with original system gives final closed-loop system

$$\dot{\mathbf{z}} = (\mathbf{A_z} - \mathbf{B_z}\,\mathbf{K_z})\mathbf{z} + \mathbf{B_z}r = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 20 & -10 & 1 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \mathbf{C_z}\,\mathbf{z} = [-1 \;\; 1 \;\; 0]\mathbf{z}$$

Observers

- The state feedback controllers we have been using only work if we have access to all of the system states.
- However, due to cost, accuracy, or availability, we may not always have the means to measure all state variables.
- When this is the case, we can estimate the states and feed the estimated states to the controller instead.
- We will use an observer (also called an estimator) to calculate the inaccessible plant state variables.

Observers - II

- We will base our observer on our plant model with output feedback to converge on the current state of system given that actual initial conditions of plant are unkown.

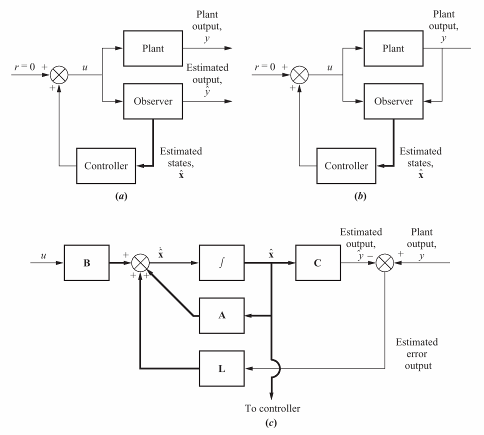


Figure 12.11.

Observer Design

- Assume a plant

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,u \qquad (26)$$
$$y = \mathbf{C}\,\mathbf{x}$$

- and observer

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\,\hat{\mathbf{x}} + \mathbf{B}\,u \qquad (27)$$
$$\hat{y} = \mathbf{C}\,\hat{\mathbf{x}}$$

- If we subtract equation 27 from equation 26, we get

$$\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) \qquad (28)$$
$$y - \hat{y} = \mathbf{C}\,(\mathbf{x} - \hat{\mathbf{x}})$$

- We have a system that will drive the difference to zero, but at the same rate as the original systems' transient response.
- This means the convergence rate of observer will be too slow to be used as input to the controller.

Observer Design - II

- To increase speed of convergence, we can feed back $y - \hat{y}$ to $\dot{\hat{\mathbf{x}}}$, as shown in Figure (c) below.

- This feedback will allow us to design a transient response for the observer that is much faster than that of the original system.

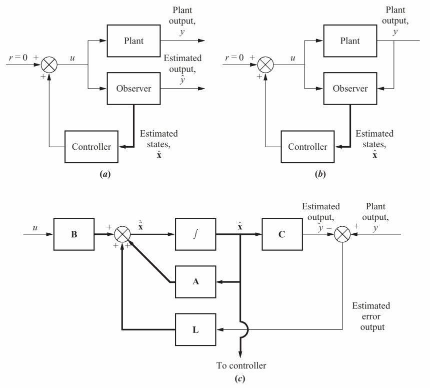


Figure 12.11.

Observer Canonical Form

- For designing state feedback controllers, systems in phase-variable form made things easier.
- For designing observers, we want systems in observer canonical form.
- Consider system below:

$$G(s) = \frac{C(s)}{R(s)} = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24} \quad (29)$$

- Now, divide all terms by highest power of s, s^3, giving:

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{s} + \frac{7}{s^2} + \frac{2}{s^3}}{1 + \frac{9}{s} + \frac{26}{s^2} + \frac{24}{s^3}} \quad (30)$$

Observer Canonical Form II

- Cross multiplying gives:

$$\left[\frac{1}{s} + \frac{7}{s^2} + \frac{2}{s^3}\right] R(s) = \left[1 + \frac{9}{s} + \frac{26}{s^2} + \frac{24}{s^3}\right] C(s) \qquad (31)$$

- Collecting terms gives:

$$C(s) = \frac{1}{s}[R(s) - 9C(s)] + \frac{1}{s^2}[7R(s) - 26C(s)] + \frac{1}{s^3}[2R(s) - 24C(s)] \qquad (32)$$

- We can rewrite this as:

$$C(s) = \frac{1}{s}[\,[R(s) - 9C(s)] + \frac{1}{s}([7R(s) - 26C(s)] + \frac{1}{s}[2R(s) - 24C(s)])] \qquad (33)$$

Observer Canonical Form III

$$C(s) = \frac{1}{s}[\,[R(s) - 9C(s)] + \frac{1}{s}([7R(s) - 26C(s)] + \frac{1}{s}[2R(s) - 24C(s)])]$$

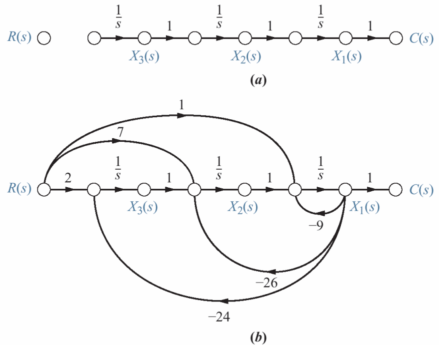


Figure 5.28.

Observer Canonical Form IV

- From signal-flow graph, we can derive state-space equations:

$$\dot{\mathbf{x}} = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} r \qquad (34)$$

$$y = [1 \ \ 0 \ \ 0]\mathbf{x}$$

- Similar form as phase-variable.
  1. Output matrix, $\mathbf{C}$, always as shown.
  2. Negate the coefficients of denominator make up left column of $\mathbf{A}$ matrix.
  3. Coefficients of numerator make up matrix $\mathbf{B}$.

$$G(s) = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

Observer Feedback e.g.

- Diagram shows plant in observer canonical form with output error feedback.

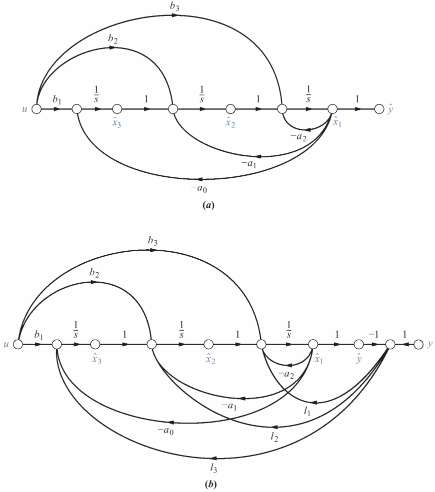


Figure 12.12.

Observer Design - Canonical Form

- From figure 12.11(c), we can derive state-space equations:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\,\hat{\mathbf{x}} + \mathbf{B}\,u + \mathbf{L}(y - \hat{y}) \tag{35}$$
$$\hat{y} = \mathbf{C}\,\hat{\mathbf{x}}$$

- Subtracting these from the equations for the plant gives:

$$(\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}) = \mathbf{A}\,(\mathbf{x} - \hat{\mathbf{x}}) - \mathbf{L}(y - \hat{y}) \tag{36}$$
$$(y - \hat{y}) = \mathbf{C}\,(\mathbf{x} - \hat{\mathbf{x}}) \tag{37}$$

- Substituting Equation 37 into 36 gives:

$$(\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}) = (\mathbf{A} - \mathbf{LC})(\mathbf{x} - \hat{\mathbf{x}}) \tag{38}$$
$$(y - \hat{y}) = \mathbf{C}\,(\mathbf{x} - \hat{\mathbf{x}}) \tag{39}$$

- If we take $e_x = (\mathbf{x} - \hat{\mathbf{x}})$ as our state variable, we see the error will go to zero as long as the eigenvalues are all in left half plane.

Observer Design - Canonical Form II

- Goal is to place the roots of characteristic equation below to get desired response.

$$\det[\lambda(\mathbf{I} - (\mathbf{A} - \mathbf{LC})] = 0 \tag{40}$$

- First, we note that for a plant in observer canonical form, $\mathbf{A} - \mathbf{LC}$, is of the form:

$$\mathbf{A} - \mathbf{LC} = \begin{bmatrix} -a_{n-1} & 1 & 0 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_{n-1} \\ l_n \end{bmatrix} [1 \;\; 0 \;\; \cdots 0] \tag{41}$$

Observer Design - Canonical Form III

- Simplifying gives

$$\mathbf{A}-\mathbf{LC} = \begin{bmatrix} -(a_{n-1}+l_1) & 1 & 0 & 0 & \cdots & 0 \\ -(a_{n-2}+l_2) & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -(a_1+l_{n-1}) & 0 & 0 & 0 & \cdots & 1 \\ -(a_0+l_n) & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (42)$$

- Our characteristic equation for $\mathbf{A}-\mathbf{LC}$ is thus

$$s^n + (a_{n-1}+l_1)s^{n-1} + (a_{n-2}+l_2)s^{n-2} + \cdots + (a_1+l_{n-1})s + (a_0+l_n) = 0 \qquad (43)$$

  We then select our poles to give desired respond giving desired characteristic equation

$$s^n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \cdots + d_1 s + d_0 = 0 \qquad (44)$$

- equating coefficients and solving for l_i, we get:

$$l_i = d_{n-i} - a_{n-i} \ \text{ for } \ i = 1, 2, \ldots n$$

Observer Design - Canonical Form e.g.

- Design an observer for plant below. The observer should respond 10 times faster than closed-loop system with dominant poles at $s = -1 \pm j2$ (designed in earlier example).

$$G(s) = \frac{(s+4)}{(s+1)(s+2)(s+5)} = \frac{(s+4)}{s^3 + 8s^2 + 17s + 10}$$

- Writing estimated plant in observer canonical form gives

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\,\hat{\mathbf{x}} + \mathbf{B}\,u \begin{bmatrix} -8 & 1 & 0 \\ -17 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} u \qquad (45)$$

$$\hat{y} = \mathbf{C}\,\hat{\mathbf{x}} = [1 \;\; 0 \;\; 0]\,\hat{\mathbf{x}}$$

- Characteristic equation for $\mathbf{A} - \mathbf{LC}$ is thus

$$s^3 + (8 + l_1)s^2 + (17 + l_2)s + (10 + l_3) \qquad (46)$$

Observer Design - Canonical Form e.g. - II

- As we want observer 10 times faster than system with dominant closed-loop poles at $s = -1 \pm j2$, we need dominant poles at $s = -10 \pm j20$.

- Choose third pole to be 10 times to the left of dominant pole to limit it's affect, gives pole at $s = -100$.

- Desired characteristic equation is thus:

$$D(s) = s^3 + 120s^2 + 2500s + 50,000 \qquad (47)$$

- Comparing coefficients for Equation above and Equation 46, gives $l_1 = 112$, $l_2 = 2483$, $l_3 = 49,990$.

Observer Response

- Response of observer with input $r(t) = 100t$, initial conditions of plant zero, and initial condition of $\mathbf{x}_1 = 0.5$.

- Top figure is with output error feedback, bottom without.

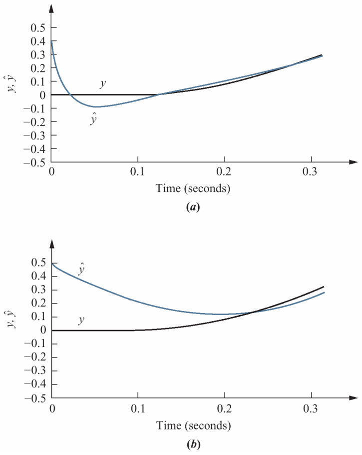


Figure 12.14.

Observervability

- To design an observer, we need to be able to deduce the current state of each state variable from the sytem output.

- If a state variable has no effect on the output, we can not determine the value of that variable from observing the output.

Definition

If initial state $\mathbf{x}(t_o)$ of system can be determined from $y(t)$ and $u(t)$ observed over a finite time interval starting at t_o, we say the system is (completely) observable. Otherwise, we say the system is unobservable.

Observervability II

- Consider system with state-space equations given below.

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,u$$
$$y = \mathbf{C}\,\mathbf{x}$$

- The observability matrix, $\mathbf{O}_M$, for the system is

$$\mathbf{O}_M = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\,\mathbf{A} \\ \vdots \\ \mathbf{C}\,\mathbf{A}^{n-1} \end{bmatrix} \tag{48}$$

- System is obervable if $\mathbf{O}_M$ is of rank n.

Alternate Approaches to Observer Design

- Observer design is very straightforward when system in observer canonical form.
- For other forms, we can still evaluate the observer and desire characteristic equation and compare coefficients, but the results typically lead to difficult calculations.
- An easier method is to transform the system into observer canonical form, place the poles, and then transform the result back into the original form.

Observer Design by Transformation

- Assume plant below is not in Observer canonical form

$$\dot{\mathbf{z}} = \mathbf{A}\,\mathbf{z} + \mathbf{B}\,u \qquad (49)$$
$$y = \mathbf{C}\,\mathbf{z}$$

- System's observability matrix is

$$\mathbf{O}_{M_z} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\,\mathbf{A} \\ \vdots \\ \mathbf{C}\,\mathbf{A}^{n-1} \end{bmatrix} \qquad (50)$$

- Assuming we can use the transform $\mathbf{z} = \mathbf{P}\,\mathbf{x}$ to transform system into observer canonical form, we get equations

$$\dot{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{A}\,\mathbf{P}\,\mathbf{x} + \mathbf{P}^{-1}\mathbf{B}\,u \qquad (51)$$
$$y = \mathbf{C}\,\mathbf{P}\,\mathbf{x}$$

Observer Design by Transformation - II

- This gives observability matrix:

$$\mathbf{O}_{M_x} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\,\mathbf{A} \\ \vdots \\ \mathbf{C}\,\mathbf{A}^{n-1} \end{bmatrix} \mathbf{P} = \mathbf{O}_{M_z}\mathbf{P} \qquad (52)$$

- Solving for $\mathbf{P}$ gives

$$\mathbf{P} = \mathbf{O}_{M_z}^{-1}\mathbf{O}_{M_x} \qquad (53)$$

- After using the observer canonical form to solve for feedback matrix $\mathbf{L}_x$, we can derive the feedback matrix for original system using the relation below:

$$\mathbf{L}_z = \mathbf{P}\,\mathbf{L}_x \qquad (54)$$

Observer Design by Transformation e.g.

- Design an observer for plant

$$G(s) = \frac{1}{(s+1)(s+2)(s+5)} = \frac{1}{s^3 + 8s^2 + 17s + 10}$$

represented in cascade form below. The desired closed-loop performance for the observer is represented by the desired characteristic equation of: $D(s) = s^3 + 120s^2 + 2500s$ $+50,000$.

$$\dot{\mathbf{z}} = \mathbf{A}\,\mathbf{z} + \mathbf{B}\,u = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad (55)$$

$$y = \mathbf{C}\,\mathbf{z} = [1 \;\; 0 \;\; 0]\mathbf{z}$$

Observer Design by Transformation e.g. - II

- The System's observability matrix is

$$\mathbf{O}_{M_z} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\,\mathbf{A} \\ \mathbf{C}\,\mathbf{A}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 25 & -7 & 1 \end{bmatrix} \tag{56}$$

- As $\det(\mathbf{O}_{M_z}) = 1 \neq 0$, the matrix has rank 3, thus the system is observable.

- Using the denominator of $G(s)$, we can construct the observer canonical form for the system.

$$\dot{\mathbf{x}} = \mathbf{A}_x\,\mathbf{x} + \mathbf{B}_x\,u \tag{57}$$
$$y = \mathbf{C}_x\,\mathbf{x}$$

with

$$\mathbf{A}_x = \begin{bmatrix} -8 & 1 & 0 \\ -17 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{C}_x = [1 \;\; 0 \;\; 0]$$

Observer Design by Transformation e.g. - III

- The observability matrix is

$$\mathbf{O}_{M_x} = \begin{bmatrix} \mathbf{C}_x \\ \mathbf{C}_x\,\mathbf{A}_x \\ \mathbf{C}_x\,\mathbf{A}_x^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 47 & -8 & 1 \end{bmatrix} \quad (58)$$

- Next step is to design an observer for the observer canonical form.

- Characteristic equation for $\mathbf{A}_x - \mathbf{L}_x\mathbf{C}_x$ is thus

$$s^3 + (8 + l_1)s^2 + (17 + l_2)s + (10 + l_3) \quad (59)$$

- Equating coefficients with the desired charactersitic equation, $D(s) = s^3 + 120s^2 + 2500s\ + 50,000$, we get

$$\mathbf{L}_x = \begin{bmatrix} 112 \\ 2483 \\ 49,990 \end{bmatrix} \quad (60)$$

Observer Design by Transformation e.g. - IV

- We now need to find $\mathbf{P}$ to transform $\mathbf{L}_x$ into $\mathbf{L}_z$.

$$\mathbf{P} = \mathbf{O}_{M_z}^{-1}\mathbf{O}_{M_x} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \tag{61}$$

and thus

$$\mathbf{L}_z = \mathbf{P}\mathbf{L}_x = \begin{bmatrix} 112 \\ 2147 \\ 47,619 \end{bmatrix} \tag{62}$$

Observer Design by Transformation e.g. - IV

- Diagram below shows original plant in cascade form, connected to the observer with output error feedback.

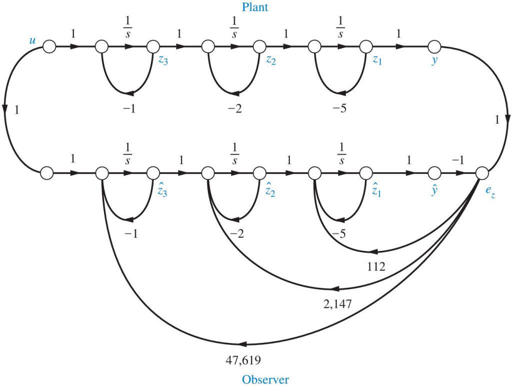


Figure 12.18.

Steady-State Error Design via Integral Control

- We now discuss how to design state space systems for steady-state error.
- To do this, we will take the controller we designed earlier, add a feedback path for the output to create error signal, and then add an integrator.
- We have added a new state variable, X_N, to the output of the new integrator, thus giving $\dot{x}_N = r - \mathbf{C}\mathbf{x}$.

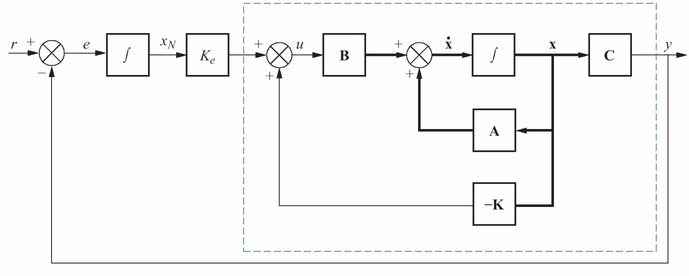


Figure 12.21.

Steady-State Error Design via Integral Control - II

- We can now write our state-space equations using augmented vectors and matrices.

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r \quad (63)$$

$$y = [\mathbf{C} \;\; 0] \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix}$$

- From diagram, we have $u = -\mathbf{K}\,\mathbf{x} + K_e x_N$. Substituting this into Equation 70 and simplifying gives

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} (\mathbf{A} - \mathbf{B}\,\mathbf{K}) & \mathbf{B}K_e \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r \quad (64)$$

$$y = [\mathbf{C} \;\; 0] \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix}$$

- We would now use the characteristic equation of the system matrix for the above system to design $\mathbf{K}$ and K_e to achieve the desired transient response.

Steady-State Error Design via Integral Control - III

- We now have another closed-loop pole we have to place that can have an effect on transient response..
- We also have to take into consideration the effect of closed-loop zeros.
- We can assume that closed-loop zeros will be in same place as the open-loop ones, but we must later verify this.
- Using this assumption, we will try to place higher order poles to cancel the zeros.

Design via Integral Control e.g.

- Consider plant below:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{65}$$

$$y = [1 \;\; 0]\mathbf{x}$$

1. Without integral control, design acontroller that gives 10% overshoot, and 0.5 second settling time. What is the steady-state error for a unit step?
2. Repeat using integral control.

- Using the required settling time and % overshoot, we calculate we need dominant closed-loop poles at $s = -8 \pm 10j$ and characteristic equation

$$s^2 + 16s + 183.1 \tag{66}$$

Design via Integral Control e.g. - II

- As plant is in phase-variable form, the characteristic equation for $\mathbf{A} - \mathbf{B}\,\mathbf{K}$ is thus:

$$s^2 + (5 + k_2)s + (3 + k_1) \tag{67}$$

- Equating coefficients and solving for gains gives $k_1 = 180.1$ and $k_2 = 11$.

- Our closed-loop plant is thus:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\,\mathbf{K})\mathbf{x} + \mathbf{B}\,r = \begin{bmatrix} 0 & 1 \\ -183.1 & -16 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \tag{68}$$

$$y = \mathbf{C}\,\mathbf{x} = [1 \;\; 0]\mathbf{x}$$

- Using equation below, we find $e_{ss} = 0.995$.

$$e_{ss} = \lim_{s \to 0} s\,R(s)[1 - \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}] \tag{69}$$

Design via Integral Control e.g. - III

- Part 2: Using equation from Slide 62, our integral-controlled plant is thus:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} (\mathbf{A} - \mathbf{B}\,\mathbf{K}) & \mathbf{B}K_e \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r \qquad (70)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \;\; k_2] \right) & \mathbf{B}K_e \\ -[1 \;\; 0] & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -(3+k_1) & -(5+k_2) & K_e \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [\mathbf{C} \;\; 0] \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix} = [1 \;\; 0 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix}$$

Design via Integral Control e.g. - IV

- We still want our dominant poles at $s = -8 \pm 10j$ to satisfy our performance requirements.
- As the open loop system has no zeros, we will assume the closed loop has none also (check later).
- We choose our third pole at $s = -100$ to minimize its effect.
- Combining the three poles gives desired characteristic equation below:

$$D(s) = s^3 + 116s^2 + 1783.1s + 18,310 \tag{71}$$

- Calculating the characteristic equation for the system matrix directly gives

$$s^3 + (5 + k_2)s^2 + (3 + k_1)s + K_e \tag{72}$$

- Comparing coefficients gives $k_1 = 1780.1$, $k_2 = 111$, $K_e = 18,310$.

Design via Integral Control e.g. - V

- This gives us a closed-loop state space representation of:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1783.1 & -116 & 18,310 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [1 \;\; 0 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix}$$

- To check is our assumption about the system's zeros was correct, we calculate the systems transfer function and find that it does not contain a zero.

$$T(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \quad (73)$$

$$= \frac{18,310}{s^3 + 116s^2 + 1783.1s + 18,310}$$

- Using equation below, we find $e_{ss} = 0$.

$$e_{ss} = \lim_{s \to 0} s\, R(s)[1 - \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}] \quad (74)$$